9.2 Power series.

A power series is basically an infinite series involving infinitely high powers of \( x \).

Ex: Known \( \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots \) is a geometric series, converges

(for \( -1 < r < 1 \)) to \( \frac{1}{1-r} \).

Can replace \( r \) by \( x \):
\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \\
= \frac{1}{1-x} \quad \text{for} \quad 1 < x < 1.
\]

Graphically:
A generic power series is written:

\[ \sum_{k=0}^{\infty} C_k (x-a)^k \]

\[ = C_0 + C_1 (x-a) + C_2 (x-a)^2 + C_3 (x-a)^3 + \ldots \]

Such a power series represents a function

\[ f(x) \]

at all \( x \) for which the series converges.

The set of all \( x \) for which a power series converges is always a single interval, called the interval of convergence.
I need this for the example that follows:

A more common version of the Ratio Test:

For any series \( \sum_{k=1}^{\infty} c_k \),

let \( r = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right| \).

Then: if \( r < 1 \), the series \( \sum_{k=1}^{\infty} c_k \) is convergent (absolutely).

\[ \ldots + 2 + 2.1 - 2.25 + \ldots \]

\[ \ldots + 1.4 \]

if \( r > 1 \), the series \( \sum_{k=1}^{\infty} c_k \) is divergent.

if \( r = 1 \), inconclusive.

\( \lim_{k \to \infty} \sqrt[k]{a_k} \)
Ex: Find the interval of convergence for the series

\[ \sum_{k=0}^{\infty} \frac{k}{3^k} (x-2)^k. \]

Note: \[ \sum = 0 + \frac{1}{3} (x-2) + \frac{2}{9} (x-2)^2 + \frac{3}{27} (x-2)^3 + \ldots \]

Fact: for any \( x \) on the inside of the interval of convergence, a power series either converges absolutely or diverges.

So we can test the series for absolute convergence, using the Ratio Test.
\[
\sum_{k=0}^{\infty} \frac{k}{3^k} (x-2)^k
\]

\[
\frac{a_{k+1}}{a_k} \sum |a_k|
\]

\[
\lim_{k \to \infty} \frac{k+1}{3^{k+1}} (x-2)^{k+1} = \lim_{k \to \infty} \frac{k+1}{3^{k+1}} (x-2)
\]

\[
= \lim_{k \to \infty} \left( \frac{k+1}{3^k} (x-2) \right) = (x-2) \lim_{k \to \infty} \left( \frac{k+1}{3^k} \right)
\]

\[
= \frac{1}{3} (x-2),
\]

So by Ratio Test, series will converge absolutely (and therefore converge) for
\[ \frac{1}{3} |x-2| < 1 \]

\[ \Rightarrow \quad 1 |x-2| < 3 \Rightarrow \quad -3 < x-2 < 3 \]

\[ \Rightarrow \quad -1 < x < 5 \]

If \( x < -1 \) or \( 5 < x \), then the limit

\[ \frac{1}{3} |x-2| \] is \( > 1 \), so series diverges.

\[ q_0 + q_1 + q_2 + \ldots \]
When $x = -1$ or $x = 5$, the limit
\[ \frac{1}{x-2} \] is 1, inconclusive.

When $x = 5$, series
\[ \sum_{k=0}^{\infty} \frac{k}{3^k} (x-2)^k \]
becomes
\[ \sum_{k=0}^{\infty} \frac{k}{3^k} 3^k = \sum_{k=0}^{\infty} k \]
divergent.

When $x = -1$, series becomes
\[ \sum_{k=0}^{\infty} \frac{k}{3^k} (-3)^k \]
\[ = \sum_{k=0}^{\infty} k(-1)^k \]
also divergent.

So the interval of convergence is $[-1, 5]$.

"Radius of convergence" $= \frac{1}{3}(\text{length of } [-1, 5]) = 3$. 
1. Abs. conv, cond. conv, or div?

\[ \sum_{k=1}^{\infty} \frac{(-1)^k \cdot k}{2k+1} \]

The term is \( \frac{(-1)^k \cdot k}{2k+1} \), which does not \( \rightarrow 0 \) as \( k \rightarrow \infty \).
Abs. conv., cond. conv., or diverges?

\[\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - 3}\]

Absolute converges is

\[\sum_{k=1}^{\infty} \frac{1}{k^2 - 3},\]

and \(\lim_{k \to \infty} \left( \frac{1}{\frac{k^2 - 3}{k^2}} \right) = 1\), and \(\sum_{k=1}^{\infty} \frac{1}{k^2}\) converges, so...

Which file would make

\[\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - 1}\]

Conditionally convergent?

A) \(f(k) = (-1)^k\)

B) \(f(k) = \frac{1}{k^2 - 1}\)

C) None of these