7.8 \( \int_0^\infty \frac{1}{x^2} \sec^2 \left( \frac{1}{x} \right) \, dx = \infty \).

\[ \int \frac{1}{x^2} \sec^2 \left( \frac{1}{x} \right) \, dx = ???
\]

Let \( u = \frac{1}{x} = x^{-1} \),

then \( du = -x^{-2} \, dx = -\frac{1}{x^2} \, dx \),

\[ = -\int \sec^2 (u) \, du \]

When \( x = \frac{1}{\pi} \), \( u = \frac{\pi}{4} \).

(bcc. \( u = \frac{x}{x} \))

When \( x \to \infty \), \( u \to 0^+ \).
\[
\int_{\frac{\pi}{4}}^{t} \sec^2(u) \, du = -\int_{\frac{\pi}{4}}^{t} \sec^2(u) \, du
\]

So:

\[
\int_{\frac{\pi}{4}}^{t} \sec^2(u) \, du = \tan(u) \bigg|_{\frac{\pi}{4}}^{t}
\]

\[
= 1 - \tan(t), \text{ so}
\]

\[
\text{(orig 5)} = \lim_{t \to 0^+} (1 - \tan(t)) = 1
\]
\[ q_2 : \lim_{a \to \infty} \int_{\frac{4}{\pi}}^{a} dx = \lim_{a \to \infty} \int_{\frac{4}{\pi}}^{a} \frac{1}{n} \sec^2 (u) du = \ldots \]

\[ \int_{0}^{\frac{\pi}{15^2}} \sec (15x) \tan (15x) \, dx \quad \left( \frac{d}{dx} (\sec(x)) = \sec(x) \tan(x) \right) \]

\[ = \frac{1}{15} \sec (15x) \left[ \frac{\pi}{30} \right]_{0}^{\frac{\pi}{15^2}} = \text{OOPS. } \sec (15x) \text{ has VA at } x = \frac{\pi}{30}. \]
\[ 50 \int_0^{\frac{\pi}{30}} \sec(15x) \tan(15x) \, dx \]

\[ = \lim_{a \to \frac{\pi}{30}} \left( \int_0^{a} \sec(15x) \tan(15x) \, dx \right) \]

\[ = \lim_{a \to \frac{\pi}{30}} \left( \frac{1}{15} \sec(15x) \right)^{a}_0 \]

\[ = \lim_{a \to \frac{\pi}{30}} \left( \frac{1}{15} \sec(15a) - \frac{1}{15} \right) = \]
\[ \lim_{a \to \frac{\pi}{30}} \left( \frac{1}{15} \cdot \frac{1}{\tan(15a)} - \frac{1}{15} \right) = +\infty \]

(DNE), (divergent)

Some trig subs:

Find \[ \int \frac{x^2}{\sqrt{25-x^2}} \, dx \]

Recall:
\[
\begin{align*}
\sqrt{a^2-x^2} & \quad \text{involved} \quad \Rightarrow \quad x = a \sin(\theta) \\
\sqrt{a^2+x^2} & \quad \text{involved} \quad \Rightarrow \quad x = a \tan(\theta) \\
\sqrt{x^2-a^2} & \quad \Rightarrow \quad x = a \sec(\theta)
\end{align*}
\]
Let \( x = 5 \sin(\theta) \):

or: \( \frac{x}{5} = \sin(\theta) \)

\[
\sqrt{25 - x^2} = \sqrt{25 - (5 \sin(\theta))^2} = 5 \cos(\theta).
\]

\[
dx = 5 \cos(\theta) \, d\theta
\]

\[
\int \frac{x^2}{\sqrt{25 - x^2}} \, dx = \int \frac{(5 \sin(\theta))^2}{5 \cos(\theta)} \cdot 5 \cos(\theta) \, d\theta
\]

\[
= \int 25 \sin^2(\theta) \, d\theta = \int 25 \cdot \frac{1}{2} (1 - \cos(2\theta)) \, d\theta
\]
\[ \frac{25}{2} \left( \theta - \frac{1}{2} \sin(2\theta) \right) + C \quad (\sin(2\theta) = 2\sin\theta\cos\theta) \]

\[ = \frac{25}{2} \left( \theta - \sin(\theta)\cos(\theta) \right) + C \]

\[ = \frac{25}{2} \left( \sin^{-1} \left( \frac{x}{5} \right) - \frac{x}{5} \cdot \frac{\sqrt{25-x^2}}{5} \right) + C \]

---

**What if** \( \int x^2 \sqrt{20-4x^2} \, dx \)?

Then: \( \sqrt{20-4x^2} = \sqrt{4(5-x^2)} \)

\[ = 2\sqrt{5-x^2} \]

So \( \int x^2 \sqrt{20-4x^2} \, dx = 2 \int x^2 \sqrt{5-x^2} \, dx \),
new let $x = \sqrt{5} \sin \theta$, etc.

Differential Equs:

$\text{Ex:}$ \quad xy' = y^2. \quad \text{Solve.}$

Separate:

$\frac{1}{y^2} y' = \frac{1}{x}$

$\int \text{B.S.} \, dx:

\int \frac{1}{y^2} \frac{dy}{dx} \, dx = \int \frac{1}{\sqrt{x}} \, dx$
\[ \Rightarrow \int y^{-2} \, dy = \int \frac{1}{x} \, dx \]
\[ \Rightarrow -y^{-1} = \ln |x| + C \]
\[ \Rightarrow \frac{1}{y} = -\ln |x| + C \]
\[ \Rightarrow y = \frac{-1}{\ln |x| + C} = \frac{-1}{\ln |1| + C} \]

If given an initial condition, like \( y(1) = 2 \),

then:
\[ 2 = \frac{-1}{\ln |1| + C} \Rightarrow 2 = -\frac{1}{C} \]
\[ \Rightarrow C = -\frac{1}{2} \]
Example of 2nd order differential eqn:

\[ x^2 y'' = y'. \]

Let's let \( y' = u \)

\[ x^2 u' = u \]

Can now separate, integrate:

\[ \frac{u'}{u} = \frac{1}{x^2} \Rightarrow \ln(u) = -x^{-1} + c \]

\[ -\frac{1}{x} + c \]

\[ u = e^{-\frac{1}{x}} + c \]

\[ u = e^{-\frac{1}{x}} \cdot e^c \]
\[ \frac{1}{x} \]

Now, \( u = k e^{-\frac{1}{x}} \).

So, \( y = \int k e^{-\frac{1}{x}} \, dx \).