Math 171 Fall 2014 Final Exam

December 15, 2014

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First Name: 

Student ID: 

Section: 

Remember to show all of your work and provide all necessary explanations for full credit. Good luck and have a good break!

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Question 1. (4 points each) Compute the following derivatives.

A. \[ \frac{d}{dx} \left( 5x^3 - 2 - \frac{3}{\sqrt{x}} + \frac{1}{x^2} \right) = \frac{d}{dx} \left( 5x^3 - \frac{x}{3} + x^{-\frac{3}{2}} \right) \]
\[ = 15x^2 - 0 - \frac{3}{2}x^{-\frac{5}{2}} - 2x^{-3} \]

B. \[ \frac{d}{d\theta} \left( \sin \theta - \cos \theta + \tan \theta - \sec \theta \right) \]
\[ = \cos \theta + \sin \theta + \sec^2 \theta - \sec \theta \tan \theta \]  
Recall that \( \sec^2 \theta \) means \( (\sec \theta)^2 \).

C. \[ \frac{d}{dx} \left( e^x + \ln x - \tan^{-1} x + \sin^{-1} x \right) \]
\[ = e^x + \frac{1}{x} - \frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}} \]

D. \[ \frac{d}{dx} \left( (x^3 + 4x^2) e^x \right) = (x^3 + 4x^2) \cdot \frac{d}{dx} (e^x) + (e^x) \cdot \frac{d}{dx} (x^3 + 4x^2) \]
\[ = (x^3 + 4x^2) e^x + e^x (3x^2 + 8x) \]

E. \[ \frac{d}{dx} \left( \frac{\sin x + \cos x}{x^2 + 1} \right) \]
\[ = \frac{(x^2 + 1) \frac{d}{dx} (\sin x + \cos x) - (\sin x + \cos x) \frac{d}{dx} (x^2 + 1)}{(x^2 + 1)^2} \]
\[ = \frac{(x^2 + 1) (\cos x - \sin x) - (\sin x + \cos x) (2x)}{(x^2 + 1)^2} \]

F. \[ \frac{d}{dx} (\ln(e^{2x} - \tan x)) = \frac{1}{e^{2x} - \tan x} \cdot \frac{d}{dx} (e^{2x} - \tan x) \]
\[ = \frac{e^{2x} \cdot 2 - \sec^2(x)}{e^{2x} - \tan(x)} \]
Question 2. (6 points each) Compute the following indefinite integrals.

A. \( \int (2x^3 - 3 + x^{-1} - x^{-3}) \, dx = \left[ \frac{2}{4} x^4 - 3x + \ln |x| - \left( -\frac{1}{4} x^{-2} \right) \right] + C \)

\[ \int x^{-1} \, dx = \ln |x| + C \]

Note:
\[ \int x^{-1} \, dx = \ln |x| + C \]

B. \( \int x^2 \sec^2(x^3) \, dx \) \quad let \ u = x^3, \ so \ du = 3x^2 \, dx \quad or \quad \frac{1}{3} \, du = x^2 \, dx \)

\[ \int \frac{1}{3} \sec^2(u) \, du = \frac{1}{3} \tan(u) + C = \frac{1}{3} \tan(x^3) + C \]

Question 3. (7 points each) Compute the following definite integrals. You are not required to simplify your answers.

A. \( \int_0^{\pi/2} (x^2 + \sin x) \, dx = \left[ \frac{1}{3} x^3 - \cos(x) \right] \bigg|_0^{\pi/2} \)

\[ = \left[ \frac{1}{3} \left( \frac{\pi}{2} \right)^3 - \cos \left( \frac{\pi}{2} \right) \right] - \left[ \frac{1}{3} (0)^3 - \cos (0) \right] = \frac{\pi^3}{24} + 1 \]

B. \( \int_1^e \frac{(\ln x)^2}{x} \, dx \) \quad let \ u = \ln(x), \ so \ du = \frac{1}{x} \, dx \).

\[ \text{when} \ x = 1, \ u = 0 \quad \text{when} \ x = e, \ u = 1 \]

\[ = \int_1^e \left( \frac{\ln x)^2}{u} \right) \, \frac{1}{du} \, du = \int_0^1 u^2 \, du = \frac{1}{3} u^3 \bigg|_0^1 = \frac{1}{3} \]
Question 4. (4 points each) Consider the function \( f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 1 \).
For reference, \( f'(x) = (x + 1)(x - 1)^2 \) and \( f''(x) = (x - 1)(3x + 1) \). For full points on each part, justification must be provided.

A. List the critical point(s) of \( f \). \( f' \) is never undefined, so crit. pts are where \( f'(x) = 0 \), which are \( x = -1, x = 1 \).

B. On what interval(s) is the function \( f \) decreasing?

\[
f'(x) = (x+1)(x-1)^2 = -o + o + \]
\[
x: \begin{array}{c}
-1 \\
1
\end{array}
\]

\( f \) decreasing on \( (-\infty, -1) \).

C. For each critical point, state whether it is a local minimum, a local maximum or neither.

By chart, \( f \) has local min at \( x = -1 \), no local extremum at \( x = 1 \).

D. On what interval(s) is \( f \) concave up?

\[
f''(x) = (x-1)(3x+1) = +o - o + \]
\[
x: \begin{array}{c}
-\frac{1}{3} \\
1
\end{array}
\]

\( f \) is concave up on \( (-\infty, -\frac{1}{3}) \) and on \( (1, \infty) \).

E. List the inflection points(s) of \( f \).

Sign of \( f'' \) changes at \( x = -\frac{1}{3} \) and at \( x = 1 \), so inflection points at \( x = -\frac{1}{3}, x = 1 \).

(The points on the graph of \( f \) are \( (1, \frac{9}{12}) \) and \( (-\frac{1}{3}, \frac{-145}{324}) \).)
Question 5. (20 points) Draw a graph of \( y = f(x) \) on the grid below that passes through the points \((-1,0), (0,1), (1,0), \) and \((2, -1)\) indicated by black dots on the grid. Your graph for \( f \) must be continuous and satisfy the following sign data for \( f' \) and \( f'' \). Also assume \( f'(0) \) and \( f''(0) \) do not exist.

<table>
<thead>
<tr>
<th>Interval</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
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<tbody>
<tr>
<td>((-\infty, -1))</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>((-1,0))</td>
<td>Positive</td>
<td></td>
</tr>
<tr>
<td>((0,1))</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>((1,2))</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>((2, \infty))</td>
<td>Positive</td>
<td></td>
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</tbody>
</table>
Question 6. (20 points) A rectangular sheet of cardboard of width $4w$ and height $h$ (in inches) can be folded into quarters and joined at the ends to make a "square tube" of volume $V = w^2h$ as drawn below:

The manufacturer of the cardboard sheet insists that $h$ be no more than 16 inches and $w$ be no more than 4 inches. What are the values of $h$ and $w$ so that the volume of the square tube is 16 in$^3$ and the quantity $P = w + \frac{1}{4}h$ is minimized?

For full credit, you must justify that you have in fact minimized, rather than maximized, the value of $P$.

Objective function is $P = w + \frac{1}{4}h$.

Constraint is $\text{vol} = 16$, or $w^2h = 16$, so $h = \frac{16}{w^2}$. So:

$$P = w + \frac{1}{4}(\frac{16}{w^2}) = w + \frac{4}{w^2}.$$  

$$\frac{dP}{dw} = 1 - \frac{8}{w^3} = \frac{w^3 - 8}{w^3}; \quad w = 2 \quad \Rightarrow$$

So absolute min $P$ value occurs when $w = 2, h = \frac{16}{2^2} = 4$.

These dimensions satisfy the requirements that $h \leq 16$ and $w \leq 4$, so I'm good.
Question 7. (5 points) Use linear approximation to estimate \( \ln(1.05) \). (NOT ON 2015 EXAM)

Linearize \( f(x) = \ln(x) \) at \( x = 1 \):

\[
f'(x) = \frac{1}{x}, \quad f'(1) = 1, \quad f(1) = 0.
\]

\[
L(x) = f(1) + f'(1) (x-1) = 0 + 1 (x-1) = x-1.
\]

This is just the tangent line to \( y = \ln(x) \) at \( x = 1 \).

So \( \ln(1.05) \approx L(1.05) = 1.05 - 1 = 0.05 \).
Question 8. (5 points) An object’s position function is \( s(t) \) and its velocity function is \( v(t) = s'(t) \). If \( s(1) = 3 \) and \( s(4) = 15 \), the mean value theorem guarantees that the velocity of the object must be what value at some time between \( t = 1 \) and \( t = 4 \)? For full credit, explicitly show the computation that produces your result.

In the Mean Value Theorem, there is a function \( f \). In this problem, \( f \) is the position function \( s \), and then \( f' \) is the velocity function \( v \). So we have:

\[
\begin{align*}
f(1) &= s(1) = 3 \\
\epsilon &= f(4) = s(4) = 15
\end{align*}
\]

\( f'(c) \), or \( v(c) \), is guaranteed to achieve this slope at some \( c \) in \((1,4)\). This slope is:

\[
\frac{s(4) - s(1)}{4 - 1} = \frac{15 - 3}{3} = 4
\]
Question 9. (10 points) Find the absolute minimum and absolute maximum values along with their locations attained by \( f(x) = x^4 - 2x^2 + 4 \) on the interval \([-1,2]\).

\([-1,2]\) is a closed interval, \( f \) is continuous on \([-1,2]\) so absolute extrema must occur at either endpoints or critical numbers.

\[ f'(x) = 4x^3 - 4x = 4x(x^2 - 1) , \text{ which is zero when } x = 0, 1, -1. \]

\[ f(-1) = 3 \quad \text{abs. min on } [-1,2] \]
\[ f(0) = 4 \]
\[ f(1) = 3 \]
\[ f(2) = 12 \quad \text{abs. max on } [-1,2]. \]

The absolute maximum is 12 occurring at \( x = \underline{2} \)

The absolute minimum is \( \underline{3} \) occurring at \( x = \underline{-1 \text{ and } 1} \)
Question 10. (4 points each) Compute the following limits. Any uses of L'Hopital's rule must be justified for full credit.

A. \[ \lim_{x \to -2} \frac{x^3 - 8}{x^2 - 3x + 2} \quad \overset{L'H}{\rightarrow} \quad \lim_{x \to -2} \left( \frac{3x^2}{2x - 3} \right) = 12. \]

B. \[ \lim_{x \to \infty} \frac{x^2 + 3x + 2}{e^x} \quad \overset{L'H}{\rightarrow} \quad \lim_{x \to \infty} \left( \frac{2x + 3}{e^x} \right) = \lim_{x \to \infty} \left( \frac{2}{e^x} \right) = 0. \]

C. \[ \lim_{x \to 0} \frac{x^3}{x^3 - 3} \quad \overset{\text{(direct sub.)}}{=} \quad 0. \]

D. \[ \lim_{t \to 0} \frac{e^t - t - \cos t}{t^2} \quad \overset{L'H}{\rightarrow} \quad \lim_{t \to 0} \left( \frac{e^t - 1 + \sin t}{2t} \right) = \lim_{t \to 0} \left( \frac{e^t + \cos t}{2} \right) = 4. \]

E. \[ \lim_{x \to 0^+} (2x)^{3x} \quad \text{If } y = (2x)^{3x}, \quad \lim_{x \to 0^+} \ln(y) = \ln((2x)^{3x}) = 3x \cdot \ln(2x). \]

So \[ \lim_{x \to 0^+} \ln(y) = \lim_{x \to 0^+} \left( \frac{3 \ln(2x)}{x} \right) = \lim_{x \to 0^+} \left( \frac{3 \ln(2^x \cdot \frac{1}{x})}{-x - 2} \right) = \lim_{x \to 0^+} (3x) = 0. \] So \( \ln(y) \to 0 \), so \( y \to 1 \).