Question 1. Find $\frac{dy}{dx}$ for each of the following.

A. \( y = \frac{2}{3x^2} - 5\tan(x) \), so
\[
y' = \left( -\frac{4}{3} \right) x^{-3} - 5\sec^2(x)
\]

B. \( y = (3x + 1)\sin^{-1}(x^3) \)
product rule, chain rule:
\[
y' = \left(3x + 1\right) \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot (3x^2) + 3\cdot\sin^{-1}(x^3)
\]

C. \( y = \ln(x + e^{4x}) \)
chain rule:
\[
y' = \frac{1}{x + e^{4x}} \cdot (1 + e^{4x} \cdot 4)
\]

D. \( y = \frac{x}{(\sqrt{x} + 2)^3} \)
quotient, chain rule:
\[
y' = \frac{(\sqrt{x} + 2)^3 \cdot \frac{1}{2} x^{-\frac{1}{2}} - x \cdot 3(\sqrt{x} + 2)^2 \cdot \frac{1}{2} x^{-\frac{1}{2}}}{(\sqrt{x} + 2)^6}
\]

E. \( y = \int_1^x \sqrt{1+t^2} \, dt \)
By Fundamental Thm. of Calc. part 1,
\[
y' = \sqrt{1+x^2}
\]
Question 2. Integrate. Simplify your numerical answers for B and D as much as possible.

A. \[ \int \left( 2\sqrt{x} + \frac{1}{x^3} \right) \, dx = \left( 2x^{\frac{4}{3}} + x^{-2} \right) \, dx \]
\[ = 2\left( \frac{4}{6}x^{\frac{4}{3}} \right) + \frac{-1}{-2}x^{-2} + C \]

B. \[ \int_{0}^{2} (5x^2 - 3x + 3) \, dx = \left[ \frac{5}{3}x^\frac{3}{2} - \frac{3}{2}x^2 + 3x \right]_0^2 = \frac{5}{3}(8) - \frac{3}{2}(4) + 6 = \frac{412}{3} \]

C. \[ \int \left( \frac{e^x}{3} + \frac{2}{5x} \right) \, dx = \int \left( \frac{1}{3}e^x + \frac{2}{5} \cdot \frac{1}{x} \right) \, dx = \frac{1}{3}e^x + \frac{2}{5} \ln |x| + C \]

D. \[ \int_{0}^{\pi/2} (4 + \sin(x)) \, dx = \left[ 4x - \cos(x) \right]_0^{\pi/2} = [4\left( \frac{\pi}{2} \right) - \cos\left( \frac{\pi}{2} \right)] - [4(0) - \cos(0)] \]
\[ = 2\pi - 0 - 0 + 1 = \frac{2\pi + 1}{1} . \]
Question 3. Integrate. Simplify your numerical answers for A and B as much as possible.

A. \( \int_0^1 \sqrt{1+3x} \, dx \)
   \( \) let \( u = 1+3x \), so \( du = 3 \, dx \), so \( \frac{1}{3} \, du = \, dx \).
   \[
   = \int_{1+2(0)}^{1+2(1)} \frac{4}{3} \sqrt{u} \, du = \frac{4}{3} \int_{1}^{4} \frac{2}{3} u^{\frac{2}{3}} \, du = \frac{2}{9} \left( 4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) = \frac{14}{9}.
   \]

B. \( \int_{1/2}^{1} \sin(\pi t) \, dt \)
   \( \) let \( u = \pi t \), so \( du = \pi \, dt \), or \( \frac{1}{\pi} \, du = \, dt \).
   \[
   = \int_{\frac{1}{2} \pi}^{\pi} \sin(u) \, du = \frac{1}{\pi} \cos(u) \bigg|_{\frac{1}{2} \pi}^{\pi} = \frac{1}{\pi} \left( \cos(\pi) - \cos\left(\frac{\pi}{2}\right) \right) = \frac{1}{\pi}.
   \]

C. \( \int x^2(4-x^3)^5 \, dx \)
   \( \) let \( u = 4-x^3 \), so \( du = -3x^2 \, dx \), so \( -\frac{1}{3} \, du = x^2 \, dx \).
   \[
   = -\frac{1}{3} \int u^5 \, du = -\frac{1}{3} \cdot \frac{1}{6} u^6 + C = -\frac{1}{18} (4-x^3)^6 + C.
   \]

D. \( \int \frac{1}{5-6x} \, dx \)
   \( \) let \( u = 5-6x \), so \( du = -6 \, dx \), so \( -\frac{1}{6} \, du = \, dx \).
   \[
   = \int \frac{1}{u} \cdot \frac{1}{6} \, du = -\frac{1}{6} \ln |u| + C = -\frac{1}{6} \ln |5-6x| + C.
   \]
Question 4. For the function \( f(x) = 4x^3 - x^4 + 2 \), do the following:

A. Find the critical points (a.k.a. critical numbers) of \( f \).

\[
f'(x) = 12x^2 - 4x^3 = 4x^2(3-x), \text{ which is never undefined, and is zero when } x = 0, x = 3.
\]

B. On what interval(s) is the function \( f \) increasing?

\[
f'(x) = 4x^2(3-x) = \frac{0 + 0}{1 - 1} \quad \text{ for increasing on } (-\infty, 0) \text{ and on } (0,3)
\]

C. For each critical number, state whether it is a local max, a local min, or neither.

From chart in B, local max at \( x = 3 \),

neither at \( x = 0 \).

D. On what interval(s) is \( f \) concave down?

\[
f''(x) = 24x - 12x^2 = 12x(2-x) = \frac{-0 + 0}{1 - 1} \quad \text{ for concave down on } (-\infty, 0) \text{ and on } (2,\infty)
\]

E. List the \( x \) value(s) for all the inflection points of \( f \).

sign of \( f'' \) changes at \( x = 0 \) and at \( x = 2 \), so

inflection points at both \( x = 0 \) and \( x = 2 \).
Question 5. Sketch the graph of a function $f$ that satisfies all of the conditions listed below.

- $f$ is continuous and differentiable on its entire domain, which is $(-\infty, 6)$
- $\lim_{x \to 6^-} f(x) = -\infty$
- $\lim_{x \to -\infty} f(x) = 0$
- $f(4) = 3$, $f'(4) = 0$
- $f''(x) > 0$ for $x < 2$ conc. up
- $f''(x) < 0$ for $2 < x < 6$ conc. down

Question 6. A box-shaped glass aquarium (with no top) is to be made so that its two smaller sides are square ($y$ inches by $y$ inches). The aquarium must be built from 5000 square inches of glass. What should $x$ and $y$ be in order to maximize the volume held by the tank? (Note that we are ignoring the thickness of the glass; you can treat it as a generic box, as in our text).

$$V_{\text{ol}} = V = x \cdot y^2 = x y^2.$$  

Total area = 5000, so

$$2y^2 + 3xy = 5000,$$

so

$$x = \frac{5000 - 2y^2}{3y} = \frac{5000}{3y} - \frac{2y}{3}.$$  

$$V = \left(\frac{5000}{3y} - \frac{2y}{3}\right) y^2 = \frac{5000}{3} y - \frac{2}{3} y^3.$$  

$$V' = \frac{5000}{3} - 2y^2 = 2\left(\frac{2500}{3} - y^2\right): y' + 0$$

$$y: 0 \quad \sqrt{\frac{2500}{3}}$$

So max. $V$ happens when

$$y = \sqrt{\frac{2500}{3}}, \quad x = \frac{5000 - 2\left(\frac{2500}{3}\right)}{3\sqrt{\frac{2500}{3}}}$$

(simplifies to $y = \frac{50\sqrt{3}}{3}, \quad x = \frac{200\sqrt{3}}{9}$).
Question 7. Find the exact value of the area of the shaded region below.

\[ \text{Area under } y = x^2 \text{ from } x = 1 \text{ to } x = 2 \text{ is} \]
\[ \int_1^2 x^2 \, dx = \frac{1}{3} x^3 \bigg|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \]

Question 8. Explain what the Mean Value Theorem tells us about the function \( f(x) = \frac{1}{x^2} \) on the interval \([1, 4]\).

The end-to-end slope is
\[ \frac{f(4) - f(1)}{4 - 1} = \frac{1/16 - 1/1}{3} = -\frac{15/16}{3} = -\frac{5}{16}. \]

MVT says that at some \( x \) value \( c \) between 1 and 4,
the slope of \( f'(x) = \frac{1}{x^2} \) will be \(-\frac{5}{16}\).

(The \( c \) value can be solved for:
\[ f'(x) = -\frac{2}{x^3} = -\frac{5}{16} \]
\[ \Rightarrow x^3 = \frac{16}{5} \Rightarrow x = \frac{32}{5} \Rightarrow x = \frac{8\sqrt{2}}{5} \] )
Question 9. Find the absolute maximum value of \( f(x) = 5 - 2x - \frac{8}{x} \) on the interval \( 0 < x < \infty \). For full credit, you must explain how you know that you've found the maximum (not the minimum) value.

\[
\begin{align*}
  f'(x) &= -2 + \frac{8}{x^2} \\
  &= \frac{-2x^2 + 8}{x^2} \\
  &= \frac{-2(x^2-4)}{x^2} \\

table: \begin{array}{c|c|c|c}
  x & 0 & 2 & \\
  f'(x) & + & 0 & - \\
\end{array}
\end{align*}
\]

The table shows that the absolute maximum value of \( f(x) \) for \( x > 0 \) is

\[
f(2) = 5 - 2(2) - \frac{8}{2} = 5 - 4 - 4 = -3.
\]
Question 10. Compute the following limits. Use $\infty$ or $-\infty$ as an answer if appropriate. Any uses of L'Hopital's rule must be justified for full credit.

A. \[ \lim_{x \to -\infty} \frac{x + x^3 + x^5 + x^8}{3x^5 - 9x^8} = \lim_{x \to -\infty} \left( \frac{x}{-9x^8} \right) = \lim_{x \to -\infty} \left( -\frac{1}{9} \right) = \left( \frac{-1}{9} \right) \]

B. \[ \lim_{x \to 3^-} \frac{x}{9 - x^2} = \infty \]

C. \[ \lim_{x \to 2} \frac{x - 2}{\ln(4x - 7)} = \lim_{x \to 2} \left( \frac{1}{4x - 7} \right) = \left( \frac{1}{4} \right) \]

D. \[ \lim_{h \to 0} \frac{(5 + h)^2 - 25}{h} = \lim_{h \to 0} \left( \frac{2(5+h)}{1} \right) = 10 \]

\[ a = \lim_{h \to 0} \left( \frac{(5 + 10h + h^2 - 25)}{h} \right) = \lim_{h \to 0} \left( \frac{h(10+h)}{h} \right) = 10. \]

E. \[ \lim_{x \to 0^+} \left( \frac{1 + \frac{1}{x} \right)^x = 1 + \frac{1}{x} \] Let \( y = \left( 1 + \frac{1}{x} \right)^x \). Then \( \ln(y) = x \ln(1 + \frac{1}{x}) = \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}}. \)

So \[ \lim_{x \to 0^+} \ln(y) = \lim_{x \to 0^+} \left( \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} \right) = \lim_{x \to 0^+} \left( \frac{\frac{1}{x} \cdot \left( -\frac{1}{x^2} \right)}{x^2} \right) = \lim_{x \to 0^+} \left( \frac{1}{x} \right) = 0. \]

So \( y \to 1 \), and so \( y \to 1 \).
Question 11. Answer the following as best you can. Showing work is not required here, but it may gain you partial credit if your answer is wrong.

A. How many inflection points are there on the graph of \( y = x^3 \)? \( \text{one} \)
\[
y'' = 6x : \quad \begin{array}{c}
- \frac{0^+}{0^-} \\
\end{array}
\]

Define \( g(x) = \int_0^x f(t) \, dt \), where the graph of \( f(t) \) is shown on the right.

B. Find the values: \( g(1) = \sqrt{1.5} \quad g(5) = -2.5 \)

C. \( g \) has a local maximum at \( x = \frac{2}{3} \).

D. As a sphere changes size over time \( t \), its volume and radius are related by \( V = \frac{4}{3} \pi r^3 \).

Give an equation which relates \( dV/dt \) to \( dr/dt \): \( \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \)

\( \Rightarrow \) take \( \frac{d}{dt} \) of both sides

\[
\frac{dV}{dt} = \frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right) = \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt}
\]

E. If \( h(x) = \int_3^x \frac{\tan(t)}{t^2} \, dt \), then \( h'(x) = \frac{\tan(x)}{x^2} \)

F. If \( \int_2^6 f(x) \, dx = -8 \) and \( \int_6^8 f(x) \, dx = 5 \), then \( \int_2^8 f(x) \, dx = -3 \)

\[
\int_2^8 = \int_2^6 + \int_6^8 = -8 + 5 = -3
\]