M171 Fall 2014 Exam 2

2014-10-15

Last Name: ________________________________

First Name: ________________________________

Student ID: ________________________________

Section No.: ________________________________

Remember to show all of your work and provide all necessary explanations for full credit! Good luck!

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Question 1: (15 pts)

A. (5 pts) In the box, state the limit definition of derivative $f'(x)$

$$f'(x) = \lim_{h \to 0} \left( \frac{f(x+h) - f(x)}{h} \right)$$

B. (10 pts) Using this definition, show that $\frac{d}{dx} (3x^2 + 1) = 6x$.

$$f'(x) = \lim_{h \to 0} \left( \frac{\left[3(x+h)^2 + 1\right] - \left[3x^2 + 1\right]}{h} \right)$$

$$= \lim_{h \to 0} \left( \frac{3(x^2 + 2xh + h^2) - 3x^2}{h} \right)$$

$$= \lim_{h \to 0} \left( \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} \right)$$

$$= \lim_{h \to 0} \left( \frac{6xh + 3h^2}{h} \right) = \lim_{h \to 0} \left( \frac{6x + 3h}{1} \right)$$

$$= \lim_{h \to 0} (6x + 3h) = 6x + 3(0) = 6x.$$
Question 2: (20 points) Find the derivative of the following functions:

A. (5 pts) $f(x) = 3x^4 - 2x + 5$

$$f'(x) = 12x^3 - 2$$

**power rule**

B. (5 pts) $y = \frac{2-5x^2}{\sec x}$

$$y' = \frac{\sec(x)(-10x) - (2-5x^2)\cdot\sec(x)\cdot\tan(x)}{\sec^2(x)}$$

**Quotient Rule**

C. (5 pts) $g(x) = (x + 3x^2) \ln x$

$$g'(x) = (x+3x^2) \cdot \frac{1}{x} + \ln(x) (1+6x)$$

**product rule**

D. (5 pts) $s(t) = t - \frac{3}{t^2} + e^{2t}$

$$s'(t) = 1 + 6t^{-3} + e^{2t} \cdot 2$$

**power/chain rules**
Question 3: (21 points) Find the derivative of the following functions:

A. (7 pts) \( r(\theta) = \sin(\tan \theta) \)

\[ r'(\theta) = \frac{\cos(\tan(\theta)) \cdot \sec^2(\theta)}{\sec^2(\theta)} \]

chain rule

B. (7 pts) \( y = [\cos(2x^4)]^5 \)

\[ y' = 5[\cos(2x^4)]^4 \cdot (-\sin(2x^4)) \cdot 8x^3 \]

\[ = -40x^3 \sin(2x^4) [\cos(2x^4)]^4. \]

power/chain rules

C. (7 pts) \( f(x) = \sin^{-1}(x^2 + 4x) \)

\[ f'(x) = \frac{1}{\sqrt{1 - (x^2 + 4x)^2}} \cdot (2x + 4) \]

\[ = \frac{2x + 4}{\sqrt{1 - (x^2 + 4x)^2}} \]

\( \sin^{-1} \), chain rules
Question 4: (17 pts) Use implicit differentiation to find \( \frac{dy}{dx} \).

A: (8 pts) \( x^3 + 2y^2 = 5 \)

Treat \( y \) as \( f(x) \), take \( \frac{d}{dx} \), get

\[
3x^2 + 4y \cdot y' = 0.
\]

Solve for \( y' \):

\[
y' = \frac{-3x^2}{4y}.
\]

B: (9 pts) \( xy = e^y \)

Treat \( y \) as \( f(x) \), take \( \frac{d}{dx} \), get

\[
x \cdot y' + y = e^y \cdot y'.
\]

\[
\Rightarrow y = e^y \cdot y' - x \cdot y'.
\]

\[
\Rightarrow y = y' \left[ e^y - x \right].
\]

\[
\Rightarrow \frac{y}{e^y - x} = y'.
\]
Question 5: (12 pts) Use logarithmic differentiation to determine \( \frac{dy}{dx} \).

\[ y = x^{(x^2)} \]

Take \( \ln \) of both sides:

\[ \ln(y) = \ln(x^{(x^2)}) \]

Rewrite:

\[ \ln(y) = x^2 \ln(x) \]

Take \( \frac{d}{dx} \):

\[ \frac{1}{y} \cdot y' = x^2 \cdot \frac{1}{x} + 2x \cdot \ln(x) \]

Solve for \( y' \):

\[ y' = [x + 2x \ln(x)] \cdot y \]

Replace \( y \):

\[ y' = [x + 2x \ln(x)] x^{(x^2)} \]
Question 6: (15 pts) Two runners start moving from the same point. One runs south at 8 mi/h and the other runs west at 6 mi/h. At what rate is the distance between the runners increasing 2 hours later?

Moment of interest:

\[
\sqrt{12^2 + 16^2} = \sqrt{4^2 (3^2 + 4^2)} = 4 \cdot 5 = 20
\]

General scenario:

\[x^2 + y^2 = h^2\]

\[
\frac{dh}{dt} = \frac{2x \frac{dx}{dt} + 2y \frac{dy}{dt}}{2h} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{h}
\]

At moment of interest, rate relation becomes

\[(12)(6) + (16)(8) = (20) h^2\]

So \[h = \frac{72 + 128}{20} = \frac{300}{20} = 15 \text{ mi/hr}.
\]