SHOW APPROPRIATE WORK or EXPLANATION on each problem (except #4) for full credit. Box or circle your final answers. Calculators/note sheets are NOT allowed. Numbers in the [ ] indicate what each problem is worth. Check the board; there may be useful info up there. Potentially useful formulas:

\[
\begin{align*}
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2(x) &= \frac{1}{2}(1 + \cos(2x)) \\
\end{align*}
\]

For a sphere: Volume = \frac{4}{3}\pi(\text{radius})^3, Surface area = 4\pi(\text{radius})^2

For a circle: Area = \pi(\text{radius})^2, Circumference = 2\pi(\text{radius})

For a triangle: Area = \frac{1}{2}(\text{base})(\text{height})

For a rectangle: Area = (\text{base})(\text{height})

1. [6 each] Find each limit. If you use L’Hospital’s Rule, you must say so, and you should indicate how you know that the limit is indeterminate.

a) \(\lim_{x \to 0} \frac{e^x - e^{3x}}{4x} \to 0 \quad \text{L'Hopital's Rule} \quad \lim_{x \to 0} \left( \frac{e^x - 3e^{3x}}{4} \right) = \frac{1 - 3}{4} = \frac{-2}{4} = \frac{-1}{2}\)

b) \(\lim_{x \to 2^-} \frac{x}{e^{2-x}} \to \infty \quad \text{as} \quad x \to 2^- \quad \Rightarrow \quad e^{\frac{x}{2-x}} \to \infty\)

c) \(\lim_{x \to \infty} \frac{\ln(x^2 + 3)}{x} \to \infty \quad \text{L'Hopital's Rule} \quad \lim_{x \to \infty} \left( \frac{\frac{2x}{x^2 + 3}}{1} \right) = \infty\)
2. [6 each] Find \( \frac{dy}{dx} \) for each function. Algebraic simplification is not necessary.

a) \( y = \tan^{-1}(x^3 + 2) \)
\[ y' = \frac{1}{1 + (x^3 + 2)^2} \cdot 3x^2 \]

b) \( y = e^{x^2-4x} \sin^{-1}(x) \)
\[ y' = e^{x^2-4x} \cdot \frac{2x-4}{\sqrt{1-x^2}} + e^{x^2-4x} \cdot (2x-4) \sin^{-1}(x) \]

c) \( y = (\ln(x) + \cosh(x))^{-2} \)
\[ y' = -2 \left( \ln(x) + \cosh(x) \right)^{-3} \left( \frac{1}{x} + \sinh(x) \right) \]

3. [10] Find the linear approximation of the function \( f(x) = \sqrt{x} \) at \( a = 64 \) and use it to approximate the number \( \sqrt{62} \).

\[ f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(64) = \frac{1}{16}, \quad f(64) = 8, \quad 50 \]
\[ L(x) = 8 + \frac{1}{16}(x-64) \]
\[ \sqrt{62} \approx L(62) = 8 + \frac{1}{16}(-2) = \frac{77}{8} \]
4. [2 per blank] Correctly complete the following statements (no work necessary):

i) \( \lim_{x \to 0^+} (\ln(x)) = -\infty \).

ii) The domain of \( \arctan(x) \) is \((-\infty, \infty)\).

iii) The range of \( \arcsin(x) \) is \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

iv) Find the exact value: \( \sin^{-1}(\cos \left( \frac{3\pi}{4} \right)) = \left\{ \begin{array}{c}
\frac{\pi}{4} \\
\end{array} \right\} \).

v) True or False:

A) \( f(x) = \frac{\sin(\ln(x))}{\sqrt{e^x + 5}} \) has an absolute minimum value on the interval \( 1 \leq x \leq 100 \) \( \text{True} \).

B) \( f(x) = e^x \) has an absolute maximum value on the interval \( 0 \leq x < 1 \) \( \text{False} \).

5. [10] Solve one of the following two problems. If you work on both, the better effort will be counted.

<table>
<thead>
<tr>
<th>Find an expression for ( \frac{dy}{dx} ) for the curve ( 4x + y^3 = \cos(xy) ) (hint: implicit differentiation)</th>
<th>Find the derivative ( \frac{dy}{dx} ) for the curve ( y = x^{\sin(x)} ) (hint: logarithmic differentiation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4y + 3y^2 y' = -y \sin(xy)(xy') + y ) ( \frac{y'}{y} = \frac{\sin(x)}{x} \frac{\ln(x)}{\ln(x)} + \cos(x) \ln(x) )</td>
<td></td>
</tr>
<tr>
<td>( 3y^2 y' + \sin(xy)xy' = -y\sin(xy) ) ( y' = \frac{-y \sin(xy) - 4}{3y^2 + x\sin(xy)} ) ( y' = \left[ \frac{\sin(x)}{x} + \cos(x) \ln(x) \right] x^{\sin(x)} ).</td>
<td></td>
</tr>
</tbody>
</table>
6. [10] For \( f(x) = \frac{x^2}{x-2} \), show that the derivative is \( f'(x) = \frac{x^2 - 4x}{(x-2)^2} \), and then answer the questions below by analyzing the sign of \( f'(x) \).

- Interval(s) where \( f \) is increasing: \( (-\infty, 0) \cup (4, \infty) \)
- \( x \) value(s) where \( f \) has local maximum: \( x = 0 \)

\[
\begin{align*}
f'(x) &= \frac{(x-2)(2x) - x^2(1)}{(x-2)^2} = \frac{2x^2 - 4x - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} \\
&= \frac{x(x-4)}{(x-2)^2} \quad \text{sign chart:} \quad + - - \quad 0 \quad - \\
&\quad \text{intervals:} \quad (-\infty, 0) \cup (4, \infty)
\end{align*}
\]

7. [10] Find all intervals on which the function \( f(x) = 5 + 4x^3 - x^4 \) is concave up. (0, 2)

\[
\begin{align*}
f'(x) &= 12x^2 - 4x^3 \\
f''(x) &= 24x - 12x^2 = 12x(2-x) \quad \text{critical points:} \quad 0, \frac{1}{2}
\end{align*}
\]
8. [10] Solve one of the following two problems. If you work on both, you will be credited with the better effort. You may leave your answer “messy”. Include the correct units with your answer.

At 9:00 AM, ship A is 6 kilometers directly south of ship B. Ship A is sailing directly east at 8 kilometers per hour, and ship B is sailing directly south at 6 kilometers per hour. How fast is the distance between the ships changing at 9:30 AM?

The base, height and area of a rectangle are changing with time. At a certain moment, the area of the rectangle is 60 ft², the base length is 12 ft, the height is increasing at 3 ft per second, and the base is decreasing at 2 ft per second. At that same moment, what is the rate of change of the rectangle’s area?

\[
\begin{align*}
\text{At 9:00 AM:} & \quad (x, y) = (6, 0) \\
\text{Ship A:} & \quad \frac{dx}{dt} = 8 \text{ km/hr} \\
\text{Ship B:} & \quad \frac{dy}{dt} = 6 \text{ km/hr}
\end{align*}
\]

\[
\begin{align*}
\text{Distance between ships:} & \quad \sqrt{x^2 + y^2} \\
\text{Rate of change:} & \quad \frac{d}{dt} \sqrt{x^2 + y^2}
\end{align*}
\]

\[
\begin{align*}
\text{At 9:30 AM:} & \quad d = \sqrt{6^2 + 9^2} = 11.31 \text{ km}
\end{align*}
\]

\[
\begin{align*}
\text{Area of rectangle:} & \quad A = bh \\
\text{Base:} & \quad b = 12 \text{ ft} \\
\text{Height:} & \quad h = 5 \text{ ft} \\
\text{Rate of change of area:} & \quad \frac{dA}{dt} = b \frac{dh}{dt} + h \frac{db}{dt}
\end{align*}
\]

\[
\begin{align*}
\text{At moment:} & \quad A = 60 \text{ ft}^2 \\
\text{Base:} & \quad b = 12 \text{ ft} \\
\text{Height:} & \quad h = 5 \text{ ft} \\
\text{Rate of change of base:} & \quad \frac{db}{dt} = -2 \text{ ft/sec} \\
\text{Rate of change of height:} & \quad \frac{dh}{dt} = 3 \text{ ft/sec}
\end{align*}
\]

\[
\begin{align*}
\text{Rate of change of area:} & \quad \frac{dA}{dt} = (12)(3) + (5)(-2) = 26 \text{ ft}^2/\text{sec}
\end{align*}
\]