An Introduction to Stochastic Calculus

Haijun Li

lih@math.wsu.edu
Department of Mathematics
Washington State University

Week 7
Outline

1. The Itô Stochastic Integrals
   - Simple Processes
   - The General Itô Stochastic Integral
Simple Processes

Let $B = (B_t, t \geq 0)$ denote Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ denote the corresponding natural filtration. Consider a partition on $[0, T]$:

$$\tau_n : 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = T.$$
Simple Processes

Let $B = (B_t, t \geq 0)$ denote Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ denote the corresponding natural filtration. Consider a partition on $[0, T]$:

$$\tau_n : 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = T.$$ 

The stochastic process $C = (C_t, t \in [0, T])$ is said to be simple if there exists a sequence $(Z_i, i = 1, \ldots, n)$ of random variables such that

$$C_t = \sum_{i=1}^{n} Z_i I\{t_{i-1} \leq t < t_i\} + Z_n I\{t_n = T\}.$$
Let $B = (B_t, t \geq 0)$ denote Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ denote the corresponding natural filtration. Consider a partition on $[0, T]$: 

$$
\tau_n : 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = T.
$$

The stochastic process $C = (C_t, t \in [0, T])$ is said to be simple if there exists a sequence $(Z_i, i = 1, \ldots, n)$ of random variables such that 

- $(Z_i, i = 1, \ldots, n)$ is adapted to $(\mathcal{F}_{t_i}, 0 \leq i \leq n)$, i.e., $Z_i$ is a function of $(B_s, s \leq t_{i-1})$ and $EZ_i^2 < \infty$, $1 \leq i \leq n$. 

\[C_t = \sum_{i=1}^{n} Z_i I\{t_{i-1} \leq t < t_i\} + Z_n I\{t_n = T\}\]
Simple Processes

Let \( B = (B_t, t \geq 0) \) denote Brownian motion and \( \mathcal{F}_t = \sigma(B_s, s \leq t) \) denote the corresponding natural filtration. Consider a partition on \([0, T]\): 

\[ \tau_n : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T. \]

The stochastic process \( C = (C_t, t \in [0, T]) \) is said to be simple if there exists a sequence \((Z_i, i = 1, \ldots, n)\) of random variables such that

- \((Z_i, i = 1, \ldots, n)\) is adapted to \((\mathcal{F}_{t_i}, 0 \leq i \leq n)\), i.e., \(Z_i\) is a function of \((B_s, s \leq t_{i-1})\) and \(EZ_i^2 < \infty, 1 \leq i \leq n\).
- \(C_t = \sum_{i=1}^{n} Z_i I\{t_{i-1} \leq t < t_i\} + Z_n I\{t = T\}\).
Simple Processes

Let $B = (B_t, t \geq 0)$ denote Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ denote the corresponding natural filtration. Consider a partition on $[0, T]$:

$$\tau_n : 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = T.$$ 

The stochastic process $C = (C_t, t \in [0, T])$ is said to be simple if there exists a sequence $(Z_i, i = 1, \ldots, n)$ of random variables such that

- $(Z_i, i = 1, \ldots, n)$ is adapted to $(\mathcal{F}_{t_i}, 0 \leq i \leq n)$, i.e., $Z_i$ is a function of $(B_s, s \leq t_{i-1})$ and $EZ_i^2 < \infty$, $1 \leq i \leq n$.
- $C_t = \sum_{i=1}^{n} Z_i I\{t_{i-1} \leq t < t_i\} + Z_n I\{t = T\}$.

Example: $f_n(t) = \sum_{i=1}^{n} \frac{i-1}{n} I_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}(t) + \frac{n-1}{n} I_{\{T\}}(t)$ on $[0, 1]$. 
Simple Processes

Let $B = (B_t, t \geq 0)$ denote Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ denote the corresponding natural filtration. Consider a partition on $[0, T]$:

$$\tau_n : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T.$$ 

The stochastic process $C = (C_t, t \in [0, T])$ is said to be simple if there exists a sequence $(Z_i, i = 1, \ldots, n)$ of random variables such that

- $(Z_i, i = 1, \ldots, n)$ is adapted to $(\mathcal{F}_{t_i}, 0 \leq i \leq n)$, i.e., $Z_i$ is a function of $(B_s, s \leq t_{i-1})$ and $EZ_i^2 < \infty$, $1 \leq i \leq n$.
- $C_t = \sum_{i=1}^{n} Z_i I\{t_{i-1} \leq t < t_i\} + Z_n I\{t = T\}$.

Example: $f_n(t) = \sum_{i=1}^{n} \frac{i-1}{n} I_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}(t) + \frac{n-1}{n} I_{\{1\}}(t)$ on $[0, 1]$.

Example: $C_n(t) = \sum_{i=1}^{n} B_{t_{i-1}} I_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}(t) + B_{t_{n-1}} I_{\{T\}}(t)$ on $[0, T]$. 

Note that $C_t$ is a function of Brownian motion until time $t$. 

Haijun Li
An Introduction to Stochastic Calculus
Week 7
Simple Processes

Let $B = (B_t, t \geq 0)$ denote Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ denote the corresponding natural filtration. Consider a partition on $[0, T]$:

$$\tau_n : 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = T.$$

The stochastic process $C = (C_t, t \in [0, T])$ is said to be simple if there exists a sequence $(Z_i, i = 1, \ldots, n)$ of random variables such that

- $(Z_i, i = 1, \ldots, n)$ is adapted to $(\mathcal{F}_{t_i}, 0 \leq i \leq n)$, i.e., $Z_i$ is a function of $(B_s, s \leq t_{i-1})$ and $EZ_i^2 < \infty$, $1 \leq i \leq n$.
- $C_t = \sum_{i=1}^n Z_i I\{t_{i-1} \leq t < t_i\} + Z_n I\{t = T\}$.

**Example:** $f_n(t) = \sum_{i=1}^n \frac{i-1}{n} l_{[\frac{i-1}{n}, \frac{i}{n})}(t) + \frac{n-1}{n} l_{\{T\}}(t)$ on $[0, 1]$.

**Example:** $C_n(t) = \sum_{i=1}^n B_{t_{i-1}} l_{[\frac{i-1}{n}, \frac{i}{n})}(t) + B_{t_{n-1}} l_{\{T\}}(t)$ on $[0, T]$.

Note that $C_t$ is a function of Brownian motion until time $t$. 
Itô Stochastic Integrals of Simple Processes

Define

$$\int_0^T C_s dB_s := \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^n Z_i \Delta_i B.$$
Itô Stochastic Integrals of Simple Processes

Define
\[
\int_0^T C_S dB_S := \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^n Z_i \Delta_i B.
\]

Itô Integrals of Simple Processes on \([0, t], t_{k-1} \leq t < t_k\)

\[
\int_0^t C_S dB_S := \int_0^T C_S I_{[0,t]}(s) dB_S = \sum_{i=1}^{k-1} Z_i \Delta_i B + Z_k (B_t - B_{t_{k-1}}).
\]
Itô Stochastic Integrals of Simple Processes

Define
\[
\int_0^T C_s dB_s := \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^n Z_i \Delta_i B.
\]

Itô Integrals of Simple Processes on \([0, t]\), \(t_{k-1} \leq t < t_k\)

\[
\int_0^t C_s dB_s := \int_0^T C_s I_{[0,t]}(s) dB_s = \sum_{i=1}^{k-1} Z_i \Delta_i B + Z_k (B_t - B_{t_{k-1}}).
\]

**Example:** \(\int_0^t f_n(s) dB_s = \sum_{i=1}^{k-1} \frac{i-1}{n} (B_{t_i} - B_{t_{i-1}}) + \frac{k-1}{n} (B_t - B_{t_{k-1}})\) for \(\frac{k-1}{n} \leq t < \frac{k}{n}\). Note that
\[
\lim_{n \to \infty} \int_0^t f_n(s) dB_s = \int_0^t s dB_s.
\]
Itô Stochastic Integrals of Simple Processes

Define

\[ \int_0^T C_s dB_s := \sum_{i=1}^n C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^n Z_i \Delta_i B. \]

Itô Integrals of Simple Processes on \([0, t], t_{k-1} \leq t < t_k\)

\[ \int_0^t C_s dB_s := \int_0^T C_s I_{[0,t]}(s) dB_s = \sum_{i=1}^{k-1} Z_i \Delta_i B + Z_k(B_t - B_{t_{k-1}}). \]

- **Example:** \( \int_0^t f_n(s) dB_s = \sum_{i=1}^{k-1} \frac{i-1}{n} (B_{t_i} - B_{t_{i-1}}) + \frac{k-1}{n} (B_t - B_{t_{k-1}}) \) for \( \frac{k-1}{n} \leq t < \frac{k}{n} \). Note that

\[
\lim_{n \to \infty} \int_0^t f_n(s) dB_s = \int_0^t s dB_s.
\]

- **Example:** \( \int_0^t C_n(s) dB_s = \sum_{i=1}^{k-1} B_{t_{i-1}} \Delta_i B + B_{t_{k-1}}(B_t - B_{t_{k-1}}) \) for \( t_{k-1} \leq t < t_k \).
The form of the Itô stochastic integral for simple processes very much reminds us of a martingale transform, which results in a martingale.

**A Martingale Property**

The stochastic process $l_t(C) := \int_0^t C_s dB_s$, $t \in [0, T]$, is a martingale with respect to the natural Brownian filtration $(\mathcal{F}_t, t \in [0, T])$.

- Using the isometry property, $E(||l_t(C)||) < \infty$, for all $t \in [0, T]$.
Itô Integral of a Simple Process is a Martingale

The form of the Itô stochastic integral for simple processes very much reminds us of a martingale transform, which results in a martingale.

**A Martingale Property**

The stochastic process $I_t(C) := \int_0^t C_s dB_s$, $t \in [0, T]$, is a martingale with respect to the natural Brownian filtration $(\mathcal{F}_t, t \in [0, T])$.

- Using the isometry property, $E(||I_t(C)||) < \infty$, for all $t \in [0, T]$.
- $I_t(C)$ is adapted to $(\mathcal{F}_t, t \in [0, T])$. 
Itô Integral of a Simple Process is a Martingale

The form of the Itô stochastic integral for simple processes very much reminds us of a martingale transform, which results in a martingale.

**A Martingale Property**

The stochastic process $I_t(C) := \int_0^t C_s dB_s$, $t \in [0, T]$, is a martingale with respect to the natural Brownian filtration $(\mathcal{F}_t, t \in [0, T])$.

- Using the isometry property, $E(|I_t(C)|) < \infty$, for all $t \in [0, T]$.
- $I_t(C)$ is adapted to $(\mathcal{F}_t, t \in [0, T])$.
- $E(I_t(C)|\mathcal{F}_s) = I_s(C)$, for $s < t$. 
Properties

- The Itô stochastic integral has expectation zero.
The Itô stochastic integral has expectation zero.
The Itô stochastic integral satisfies the isometry property:

\[
E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \quad t \in [0, T].
\]
Properties

- The Itô stochastic integral has expectation zero.
- The Itô stochastic integral satisfies the isometry property:
  \[ E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \quad t \in [0, T]. \]
- For any constants \( c_1 \) and \( c_2 \), and simple processes \( C^{(1)} \) and \( C^{(2)} \) on \([0, T]\),
  \[ \int_0^t (c_1 C_s^{(1)} + c_2 C_s^{(2)}) dB_s = c_1 \int_0^t C_s^{(1)} dB_s + c_2 \int_0^t C_s^{(2)} dB_s. \]
Properties

- The Itô stochastic integral has expectation zero.
- The Itô stochastic integral satisfies the isometry property:

\[ E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \quad t \in [0, T]. \]

- For any constants \( c_1 \) and \( c_2 \), and simple processes \( C^{(1)} \) and \( C^{(2)} \) on \( [0, T] \),

\[ \int_0^t (c_1 C^{(1)}_s + c_2 C^{(2)}_s) dB_s = c_1 \int_0^t C^{(1)}_s dB_s + c_2 \int_0^t C^{(2)}_s dB_s. \]

- For any \( t \in [0, T] \),

\[ \int_0^T C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s. \]
Properties

- The Itô stochastic integral has expectation zero.
- The Itô stochastic integral satisfies the isometry property:

\[ E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \quad t \in [0, T]. \]

- For any constants \( c_1 \) and \( c_2 \), and simple processes \( C^{(1)} \) and \( C^{(2)} \) on \([0, T]\),

\[ \int_0^t (c_1 C_s^{(1)} + c_2 C_s^{(2)}) dB_s = c_1 \int_0^t C_s^{(1)} dB_s + c_2 \int_0^t C_s^{(2)} dB_s. \]

- For any \( t \in [0, T] \),

\[ \int_0^t C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s. \]

- The process \( I(C) \) has continuous sample paths.
Basic Assumptions

Assumptions on the Integrand Process $C$

1. $C = (C_t, t \in [0, T])$ is adapted to Brownian motion on $[0, T]$, i.e. $C_t$ is a function of $B_s$, $s \leq t$.

2. The integral $\int_0^T EC_s^2 ds < \infty$. 
Basic Assumptions

Assumptions on the Integrand Process $C$

1. $C = (C_t, t \in [0, T])$ is adapted to Brownian motion on $[0, T]$, i.e. $C_t$ is a function of $B_s, s \leq t$.

2. The integral $\int_0^T EC_s^2 ds < \infty$.

For fixed $t$ and a given partition $\tau_n = (t_i)$ of $[0, t]$, we defined $\int_0^T C_s dB_s = \sum_{i=1}^n C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$, as the Riemann-Stieltjes sums, for a simple process $C$. 
Basic Assumptions

Assumptions on the Integrand Process \( C \)

1. \( C = (C_t, t \in [0, T]) \) is adapted to Brownian motion on \([0, T]\), i.e. \( C_t \) is a function of \( B_s, s \leq t \).

2. The integral \( \int_0^T EC_s^2 ds < \infty \).

For fixed \( t \) and a given partition \( \tau_n = (t_i) \) of \([0, t]\), we defined
\[
\int_0^T C_s dB_s = \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}),
\]
as the Riemann-Stieltjes sums, for a simple process \( C \).

Brownian motion \( B = (B_t, t \in [0, T]) \) satisfies the Assumptions.
Basic Assumptions

Assumptions on the Integrand Process $C$

1. $C = (C_t, t \in [0, T])$ is adapted to Brownian motion on $[0, T]$, i.e. $C_t$ is a function of $B_s$, $s \leq t$.

2. The integral $\int_0^T EC^2_s ds < \infty$.

- For fixed $t$ and a given partition $\tau_n = (t_i)$ of $[0, t]$, we defined $\int_0^T C_s dB_s = \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})$, as the Riemann-Stieltjes sums, for a simple process $C$.

- Brownian motion $B = (B_t, t \in [0, T])$ satisfies the Assumptions.

- Simple process $C = (C_t, t \in [0, T])$ satisfies the Assumptions.
Basic Assumptions

Assumptions on the Integrand Process $C$

1. $C = (C_t, t \in [0, T])$ is adapted to Brownian motion on $[0, T]$, i.e. $C_t$ is a function of $B_s, s \leq t$.

2. The integral $\int_0^T EC_s^2 ds < \infty$.

- For fixed $t$ and a given partition $\tau_n = (t_i)$ of $[0, t]$, we defined $\int_0^T C_s dB_s = \sum_{i=1}^n C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$, as the Riemann-Stieltjes sums, for a simple process $C$.

- Brownian motion $B = (B_t, t \in [0, T])$ satisfies the Assumptions.

- Simple process $C = (C_t, t \in [0, T])$ satisfies the Assumptions.

- Another class of admissible integrands consists of the deterministic functions $c(t)$ on $[0, T]$ with $\int_0^T c^2(t) dt < \infty$. 
Key Steps to Define Itô Integrals and Proofs

Let \( C = (C_t, t \in [0, T]) \) be a process satisfying the Assumptions.

- We need to find a sequence \( C^{(n)} = (C^{(n)}_t, t \in [0, T]) \) of simple processes such that

\[
\int_0^T E[C_s - C^{(n)}_s]^2 ds \to 0, \text{ as } \text{mesh}(\tau_n) \to 0.
\]

That is, the simple processes \( C^{(n)} \) converge in a certain mean square sense to the integrand process \( C \).
Key Steps to Define Itô Integrals and Proofs

Let $C = (C_t, t \in [0, T])$ be a process satisfying the Assumptions.

- We need to find a sequence $C^{(n)} = (C^{(n)}_t, t \in [0, T])$ of simple processes such that
  \[
  \int_0^T E[C_s - C^{(n)}_s]^2 ds \to 0, \text{ as mesh}(\tau_n) \to 0.
  \]
  That is, the simple processes $C^{(n)}$ converge in a certain mean square sense to the integrand process $C$.

- Since $C^{(n)}$ is simple we can evaluate the Itô Integrals
  \[
  I_t(C^{(n)}) = \int_0^t C^{(n)}_s dB_s \text{ for every } n \text{ and } t.
  \]
Key Steps to Define Itô Integrals and Proofs

Let \( C = (C_t, t \in [0, T]) \) be a process satisfying the Assumptions.

- We need to find a sequence \( C^{(n)} = (C^{(n)}_t, t \in [0, T]) \) of simple processes such that

\[
\int_0^T E[C_s - C^{(n)}_s]^2 ds \to 0, \text{ as mesh}(\tau_n) \to 0.
\]

That is, the simple processes \( C^{(n)} \) converge in a certain mean square sense to the integrand process \( C \).

- Since \( C^{(n)} \) is simple we can evaluate the Itô Integrals

\[
I_t(C^{(n)}) = \int_0^t C^{(n)}_s dB_s \text{ for every } n \text{ and } t.
\]

- We need to show the existence of a process \( I(C) \) on \([0, T]\) such that

\[
E \sup_{0 \leq t \leq T} [I_t(C) - I_t(C^{(n)})]^2 \to 0, \text{ as mesh}(\tau_n) \to 0.
\]

That is, to show that the sequence \((I_t(C^{(n)}))\) of Itô stochastic integrals converges in a certain mean square sense to a unique limit process.
The General Itô Stochastic Integral

**Definition**

The mean square limit $I(C)$ is called the Itô stochastic integral of $C$. It is denoted by $I_t(C) = \int_0^t C_s dB_s$, $t \in [0, T]$. For practical purposes, the following rule of thumb is helpful: The Itô stochastic integrals $I_t(C) = \int_0^t C_s dB_s$, $t \in [0, T]$, constitute a stochastic process. For a given partition $\tau_n$: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t \in [t_k-l, t_k]$, the random variable $I_t(C)$ is “close” to the Riemann-Stieltjes sum

$$
\sum_{i=1}^{k-1} C_{t_i-1} (B_{t_i} - B_{t_i-1}) + C_{t_k-1} (B_t - B_{t_k-1})
$$

and this approximation is the closer (in the mean square sense) to the value of $I_t(C)$ the more dense the partition $\tau_n$ in $[0, T]$. 

Haijun Li
An Introduction to Stochastic Calculus
Week 7 9 / 10
The General Itô Stochastic Integral

**Definition**

The mean square limit $I(C)$ is called the Itô stochastic integral of $C$. It is denoted by $I_t(C) = \int_0^t C_s dB_s$, $t \in [0, T]$.

For practical purposes, the following rule of thumb is helpful:
The General Itô Stochastic Integral

**Definition**

The mean square limit $I(C)$ is called the Itô stochastic integral of $C$. It is denoted by $I_t(C) = \int_0^t C_s dB_s, \quad t \in [0, T]$.

For practical purposes, the following rule of thumb is helpful:

The Itô stochastic integrals $I_t(C) = \int_0^t C_s dB_s, \quad t \in [0, T]$, constitute a stochastic process. For a given partition

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$$

and $t \in [t_{k-1}, t_k]$, the random variable $I_t(C)$ is “close” to the Riemann-Stieltjes sum

$$\sum_{i=1}^{k-1} C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) + C_{t_{k-1}} (B_t - B_{t_{k-1}})$$

and this approximation is the closer (in the mean square sense) to the value of $I_t(C)$ the more dense the partition $\tau_n$ in $[0, T]$. 
Properties of the General Itô Stochastic Integral

- The stochastic process $\int_0^t C_s dB_s$ is a martingale with respect to the natural Brownian filtration ($\mathcal{F}_t$, $t \in [0, T]$).
Properties of the General Itô Stochastic Integral

- The stochastic process $\int_0^t C_s dB_s$ is a martingale with respect to the natural Brownian filtration ($\mathcal{F}_t, t \in [0, T]$).
- The Itô stochastic integral has expectation zero.

For any constants $c_1$ and $c_2$, and processes $C_1$ and $C_2$ on $[0, T]$,

$$\int_0^t (c_1 C_1(s) + c_2 C_2(s)) dB_s = c_1 \int_0^t C_1(s) dB_s + c_2 \int_0^t C_2(s) dB_s.$$
Properties of the General Itô Stochastic Integral

- The stochastic process $\int_0^t C_s dB_s$ is a martingale with respect to the natural Brownian filtration ($\mathcal{F}_t, t \in [0, T]$).
- The Itô stochastic integral has expectation zero.
- The Itô stochastic integral satisfies the isometry property:

$$E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \ t \in [0, T].$$
Properties of the General Itô Stochastic Integral

The stochastic process \( \int_0^t C_s dB_s \) is a martingale with respect to the natural Brownian filtration \((\mathcal{F}_t, t \in [0, T])\).

The Itô stochastic integral has expectation zero.

The Itô stochastic integral satisfies the isometry property:

\[
E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \; t \in [0, T].
\]

For any constants \( c_1 \) and \( c_2 \), and processes \( C^{(1)} \) and \( C^{(2)} \) on \([0, T]\),

\[
\int_0^t (c_1 C_s^{(1)} + c_2 C_s^{(2)}) dB_s = c_1 \int_0^t C_s^{(1)} dB_s + c_2 \int_0^t C_s^{(2)} dB_s.
\]
Properties of the General Itô Stochastic Integral

- The stochastic process $\int_0^t C_s dB_s$ is a martingale with respect to the natural Brownian filtration $(\mathcal{F}_t, t \in [0, T])$.
- The Itô stochastic integral has expectation zero.
- The Itô stochastic integral satisfies the isometry property:
  \[ E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t E C_s^2 ds, \quad t \in [0, T]. \]

- For any constants $c_1$ and $c_2$, and processes $C^{(1)}$ and $C^{(2)}$ on $[0, T]$,
  \[ \int_0^t (c_1 C^{(1)}_s + c_2 C^{(2)}_s) dB_s = c_1 \int_0^t C^{(1)}_s dB_s + c_2 \int_0^t C^{(2)}_s dB_s. \]

- For any $t \in [0, T]$,
  \[ \int_0^T C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s. \]
Properties of the General Itô Stochastic Integral

- The stochastic process $\int_0^t C_s dB_s$ is a martingale with respect to the natural Brownian filtration $(\mathcal{F}_t, t \in [0, T])$.
- The Itô stochastic integral has expectation zero.
- The Itô stochastic integral satisfies the isometry property:
  \[
  E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t EC_s^2 ds, \quad t \in [0, T].
  \]
- For any constants $c_1$ and $c_2$, and processes $C^{(1)}$ and $C^{(2)}$ on $[0, T]$,
  \[
  \int_0^t \left( c_1 C_s^{(1)} + c_2 C_s^{(2)} \right) dB_s = c_1 \int_0^t C_s^{(1)} dB_s + c_2 \int_0^t C_s^{(2)} dB_s.
  \]
- For any $t \in [0, T]$,
  \[
  \int_0^T C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s.
  \]