1. The Black-Scholes Option Pricing Formula
   - A Short Excursion into Finance
   - What is an Option?
   - A Mathematical Formulation of the Option Pricing Problem

2. Change of Measure
   - Girsanov’s Theorem

3. Extensions and Limitations of the Model
Let $X_t$ denote the price of a risky asset (let’s call it a stock) at time $t$. 

The constant $c > 0$ is the so-called mean rate of return, and $\sigma > 0$ is the volatility.

Observe that this is a crude, first order approximation to a real price process. But people in economics believe in exponential growth and they are often happy with this model.
Let $X_t$ denote the price of a risky asset (let’s call it a stock) at time $t$.

Assume that the relative return from the asset in the period of time $[t, t + dt]$ has a linear trend $c \, dt$ which is disturbed by a stochastic noise term $\sigma dB_t$.

$$\frac{X_{t+dt} - X_t}{X_t} = cdt + \sigma dB_t, \text{ or } dX_t = cX_t dt + \sigma X_t dB_t.$$
Let $X_t$ denote the price of a risky asset (let’s call it a stock) at time $t$.

Assume that the relative return from the asset in the period of time $[t, t + dt]$ has a linear trend $c \, dt$ which is disturbed by a stochastic noise term $\sigma dB_t$.

$$
\frac{X_{t+dt} - X_t}{X_t} = cdt + \sigma dB_t, \text{ or } dX_t = cX_t \, dt + \sigma X_t dB_t.
$$

The constant $c > 0$ is the so-called mean rate of return, and $\sigma > 0$ is the volatility.
Let $X_t$ denote the price of a risky asset (let’s call it a stock) at time $t$.

Assume that the relative return from the asset in the period of time $[t, t + dt]$ has a linear trend $c \, dt$ which is disturbed by a stochastic noise term $\sigma \, dB_t$.

$$\frac{X_{t+dt} - X_t}{X_t} = c dt + \sigma \, dB_t, \text{ or } dX_t = cX_t \, dt + \sigma X_t dB_t.$$  

The constant $c > 0$ is the so-called mean rate of return, and $\sigma > 0$ is the volatility.

Observe that this is a crude, first order approximation to a real price process. But people in economics believe in exponential growth and they are often happy with this model.
Trading Strategy

- Assume that you have a non-risky asset such as a bank account, which can be called a bond. Let $\beta_t$ denote the bond yield at time $t$. 

\[ \frac{d\beta_t}{\beta_t} = r dt, \]

or

\[ \beta_t = \beta_0 e^{rt}. \]

Note again that this is an idealization since the interest rate changes over time as well.

If you have $a_t$ shares in stock and $b_t$ shares in a bond at time $t$, then your portfolio at time $t$ can be represented by $(a_t, b_t)$, $t \in [0, T]$, which is called a trading strategy.

You want to adjust your strategy according to information available to you at time $t$, as to maximize your wealth $V_t = a_t X_t + b_t \beta_t$ (the value of your portfolio) at time $t$. So, it is reasonable to assume that $a_t$ and $b_t$ are stochastic processes adapted to Brownian motion $B$. 
Trading Strategy

- Assume that you have a non-risky asset such as a bank account, which can be called a bond. Let $\beta_t$ denote the bond yield at time $t$.
- Let’s say your initial capital is $\beta_0$, which will be continuously compounded with a constant interest rate $r > 0$. That is,

$$d\beta_t = r\beta_t dt, \text{ or } \beta_t = \beta_0 e^{rt}.$$ 

Note again that this is an idealization since the interest rate changes over time as well.
Trading Strategy

- Assume that you have a non-risky asset such as a bank account, which can be called a bond. Let $\beta_t$ denote the bond yield at time $t$.
- Let’s say your initial capital is $\beta_0$, which will be continuously compounded with a constant interest rate $r > 0$. That is,

$$d\beta_t = r\beta_t dt, \quad \text{or} \quad \beta_t = \beta_0 e^{rt}.$$ 

Note again that this is an idealization since the interest rate changes over time as well.

- If you have $a_t$ shares in stock and $b_t$ shares in bond at time $t$, then your portfolio at time $t$ can be represented by $(a_t, b_t)$, $t \in [0, T]$, which is called a trading strategy.
Assume that you have a non-risky asset such as a bank account, which can be called a bond. Let $\beta_t$ denote the bond yield at time $t$.

Let’s say your initial capital is $\beta_0$, which will be continuously compounded with a constant interest rate $r > 0$. That is,

$$d\beta_t = r\beta_t dt, \text{ or } \beta_t = \beta_0 e^{rt}.$$ 

Note again that this is an idealization since the interest rate changes over time as well.

If you have $a_t$ shares in stock and $b_t$ shares in bond at time $t$, then your portfolio at time $t$ can be represented by $(a_t, b_t)$, $t \in [0, T]$, which is called a trading strategy.

You want to adjust your strategy according to information available to you at time $t$, as to maximize your wealth $V_t = a_t X_t + b_t \beta_t$ (the value of your portfolio) at time $t$. So, It is reasonable to assume that $a_t$ and $b_t$ are stochastic processes adapted to Brownian motion $B$. 
Self-Financing Condition

- $a_t$ and $b_t$ can be positive or negative. A negative value of $a_t$ means *short sale of stock* (i.e. you sell the stock at time $t$). A negative value of $b_t$ means that you *borrow* money at the bond’s riskless interest rate $r$. 

\[dV_t = a_t dX_t + b_t d\beta_t = (ca_t X_t + rb_t \beta_t) dt + \sigma a_t X_t dB_t.\]
Self-Financing Condition

- $a_t$ and $b_t$ can be positive or negative. A negative value of $a_t$ means short sale of stock (i.e. you sell the stock at time $t$). A negative value of $b_t$ means that you borrow money at the bond’s riskless interest rate $r$.
- We neglect transaction costs for operations on stock and sale for simplicity.
Self-Financing Condition

- $a_t$ and $b_t$ can be positive or negative. A negative value of $a_t$ means short sale of stock (i.e. you sell the stock at time $t$). A negative value of $b_t$ means that you borrow money at the bond’s riskless interest rate $r$.
- We neglect transaction costs for operations on stock and sale for simplicity.
- Assume that you spend no money on other purposes (such as food), i.e., you do not make your portfolio smaller by consumption.
Self-Financing Condition

- $a_t$ and $b_t$ can be positive or negative. A negative value of $a_t$ means **short sale of stock** (i.e. you sell the stock at time $t$). A negative value of $b_t$ means that you **borrow** money at the bond’s riskless interest rate $r$.

- We neglect **transaction costs** for operations on stock and sale for simplicity.

- Assume that you spend no money on other purposes (such as food), i.e., you do not make your portfolio smaller by **consumption**.

- We assume finally that your trading strategy $(a_t, b_t)$ is **self-financing**. That is, the increments of your wealth $V_t$ result only from changes of the prices $X_t$ and $\beta_t$ of your asserts:

\[
dV_t = a_t dX_t + b_t d\beta_t = (ca_t X_t + rb_t \beta_t) dt + \sigma a_t X_t dB_t.
\]

\[
V_t = V_0 + \int_0^t (ca_s X_s + rb_s \beta_s) ds + \int_0^t \sigma a_s X_s dB_s.
\]
An option at time $t = 0$ is a “ticket” which entitles you to buy one share of stock until or at time $T$, the time of maturity or time of expiration of the option.

The purchaser of a European call option is entitled to a payment of $(X_T - K) + = \max(0, X_T - K)$. We illustrate option pricing using European call options.

A put is an option to sell stock at a strike price $K$ on or until a particular date of maturity $T$. A European put option is exercised only at time of maturity with profit $(K - X_T)$, and an American put can be exercised until or at time $T$. 
An option at time $t = 0$ is a “ticket” which entitles you to buy one share of stock until or at time $T$, the time of maturity or time of expiration of the option.

If you can exercise this option (or exercise the call) at a fixed price $K$, called the exercise price or strike price of the option, only at time of maturity $T$, this is called a European call option. If you can exercise it until or at time $T$, it is called an American call option. There are many other kinds ....
An option at time $t = 0$ is a “ticket” which entitles you to buy one share of stock until or at time $T$, the time of maturity or time of expiration of the option.

If you can exercise this option (or exercise the call) at a fixed price $K$, called the exercise price or strike price of the option, only at time of maturity $T$, this is called a European call option. If you can exercise it until or at time $T$, it is called an American call option. There are many other kinds ....

The purchaser of a European call option is entitled to a payment of

$$ (X_T - K)^+ = \max(0, X_T - K). $$

We illustrate option pricing using European call options.
An option at time $t = 0$ is a “ticket” which entitles you to buy one share of stock until or at time $T$, the time of maturity or time of expiration of the option.

If you can exercise this option (or exercise the call) at a fixed price $K$, called the exercise price or strike price of the option, only at time of maturity $T$, this is called a European call option. If you can exercise it until or at time $T$, it is called an American call option. There are many other kinds ....

The purchaser of a European call option is entitled to a payment of

$$(X_T - K)^+ = \max(0, X_T - K).$$

We illustrate option pricing using European call options.

A put is an option to sell stock at a strike price $K$ on or until a particular date of maturity $T$. A European put option is exercised only at time of maturity with profit $(K - X_T)^+$, and an American put can be exercised until or at time $T$. 
Since you do not know the price $X_T$ at time $t = 0$ when you purchase the call, a natural question arises:

How much would you be willing to pay for such a ticket, i.e. what is a rational price for this option at time $t = 0$?
Option Pricing

Since you do not know the price $X_T$ at time $t = 0$ when you purchase the call, a natural question arises:

How much would you be willing to pay for such a ticket, i.e. what is a rational price for this option at time $t = 0$?

Black, Scholes and Merton responded as follows:

1. You, after investing this rational value of money in stock and bond at time $t = 0$, can manage your portfolio according to a self-financing strategy so as to yield the same payoff $(X_T - K)$ as if the option had been purchased.

2. If the option were offered at any price other than this rational value, there would be an opportunity of arbitrage, i.e. for unbounded profits without an accompanying risk of loss.
Since you do not know the price $X_T$ at time $t = 0$ when you purchase the call, a natural question arises:

How much would you be willing to pay for such a ticket, i.e. what is a rational price for this option at time $t = 0$?

Black, Scholes and Merton responded as follows:

1. You, after investing this rational value of money in stock and bond at time $t = 0$, can manage your portfolio according to a self-financing strategy so as to yield the same payoff $(X_T - K)^+$ as if the option had been purchased.
Option Pricing

Since you do not know the price $X_T$ at time $t = 0$ when you purchase the call, a natural question arises:

How much would you be willing to pay for such a ticket, i.e. what is a rational price for this option at time $t = 0$?

Black, Scholes and Merton responded as follows:

1. You, after investing this rational value of money in stock and bond at time $t = 0$, can manage your portfolio according to a self-financing strategy so as to yield the same payoff $(X_T - K)^+$ as if the option had been purchased.

2. If the option were offered at any price other than this rational value, there would be an opportunity of arbitrage, i.e. for unbounded profits without an accompanying risk of loss.
Hedging Against the Contingent Claim

Goal: Find a self-financing strategy \((a_t, b_t)\) and a wealth process \(V_t\), such that
\[
V_t = a_t X_t + b_t \beta_t = u(T - t, X_t), \quad t \in [0, T],
\]
for some smooth (a technical assumption) deterministic function \(u(t, x)\) with the terminal condition
\[
V_T = u(0, X_T) = (X_T - K)^+.
\]

That is, to hedge against the contingent claim \((X_T - K)^+\).
Hedging Against the Contingent Claim

Goal: Find a self-financing strategy \((a_t, b_t)\) and a wealth process \(V_t\), such that

\[
V_t = a_t X_t + b_t \beta_t = u(T - t, X_t), \quad t \in [0, T],
\]

for some smooth (a technical assumption) deterministic function \(u(t, x)\) with the terminal condition

\[
V_T = u(0, X_T) = (X_T - K)^+.
\]

That is, to hedge against the contingent claim \((X_T - K)^+\).

Apply the Itô lemma to the wealth process \(V_t = u(T - t, X_t)\) and we obtain an integral representation.
Hedging Against the Contingent Claim

Goal: Find a self-financing strategy \((a_t, b_t)\) and a wealth process \(V_t\), such that

\[ V_t = a_t X_t + b_t \beta_t = u(T - t, X_t), \quad t \in [0, T], \]

for some smooth (a technical assumption) deterministic function \(u(t, x)\) with the terminal condition

\[ V_T = u(0, X_T) = (X_T - K)^+. \]

That is, to hedge against the contingent claim \((X_T - K)^+\).

1. Apply the Itô lemma to the wealth process \(V_t = u(T - t, X_t)\) and we obtain an integral representation.
2. Plug \(b_t = \frac{V_t - a_t X_t}{\beta_t}\) into the self-financing condition, and we obtain another integral representation.
Hedging Against the Contingent Claim

Goal: Find a self-financing strategy \((a_t, b_t)\) and a wealth process \(V_t\), such that

\[
V_t = a_t X_t + b_t \beta_t = u(T - t, X_t), \quad t \in [0, T],
\]

for some smooth (a technical assumption) deterministic function \(u(t, x)\) with the terminal condition

\[
V_T = u(0, X_T) = (X_T - K)^+.
\]

That is, to hedge against the contingent claim \((X_T - K)^+\).

1. Apply the Itô lemma to the wealth process \(V_t = u(T - t, X_t)\) and we obtain an integral representation.
2. Plug \(b_t = \frac{V_t - a_t X_t}{\beta_t}\) into the self-financing condition, and we obtain another integral representation.
3. From these two integral representations, we derive a PDE with condition \(u(0, x) = (x - K)^+, \; x > 0\).
We obtain that

\[ 0.5 \sigma^2 x^2 u_{22}(t, x) + rxu_x(t, x) + u_1(t, x) - ru(t, x) = 0 \]

with boundary conditions

\[ u(t, 0) = 0, \quad \lim_{x \to \infty} \frac{u(t, x)}{x} = 1 \quad \forall t \in [0, T]; \quad u(0, x) = (x - K)^+ \quad \forall x \geq 0. \]
Black-Scholes PDE

We obtain that

\[ 0.5\sigma^2 x^2 u_{22}(t, x) + r x u_2(t, x) + u_1(t, x) - ru(t, x) = 0 \]

with boundary conditions

\[ u(t, 0) = 0, \quad \lim_{x \to \infty} \frac{u(t, x)}{x} = 1 \quad \forall t \in [0, T]; \quad u(0, x) = (x - K)^+ \quad \forall x \geq 0. \]

Transform the equation into a diffusion equation by using

\[ \theta = T - t, \quad y = \log(x/K) + (r - \sigma^2/2)\theta, \quad w(\theta, y) = e^{r\theta}u(t, x). \]
We obtain that

\[ 0.5\sigma^2 x^2 u_{22}(t, x) + r x u_2(t, x) + u_1(t, x) - ru(t, x) = 0 \]

with boundary conditions

\[ u(t, 0) = 0, \quad \lim_{x \to \infty} \frac{u(t, x)}{x} = 1 \quad \forall t \in [0, T]; \quad u(0, x) = (x - K)^+ \quad \forall x \geq 0. \]

Transform the equation into a diffusion equation by using

\[ \theta = T - t, \quad y = \log(x/K) + (r - \sigma^2/2)\theta, \quad w(\theta, y) = e^{r\theta} u(t, x). \]

We arrive at a heat equation

\[ \frac{\partial w}{\partial \theta} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} \]

with an initial condition \( w(0, y) = K(e^y - 1)^+ \).
We obtain that
\[ 0.5\sigma^2 x^2 u_{22}(t, x) + r x u_2(t, x) + u_1(t, x) - ru(t, x) = 0 \]
with boundary conditions
\[ u(t, 0) = 0, \lim_{x \to \infty} \frac{u(t, x)}{x} = 1 \quad \forall t \in [0, T]; u(0, x) = (x - K)^+ \quad \forall x \geq 0. \]

Transform the equation into a diffusion equation by using
\[ \theta = T - t, \quad y = \log(x/K) + (r - \sigma^2/2)\theta, \quad w(\theta, y) = e^{r\theta} u(t, x). \]

We arrive at a heat equation
\[ \frac{\partial w}{\partial \theta} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} \]
with an initial condition \( w(0, y) = K(e^y - 1)^+ \).

Use the heat kernel, we have
\[ w(\theta, y) = (2\pi\sigma^2\theta)^{1/2} \int_{-\infty}^{\infty} w(0, z)e^{-\frac{(y-z)^2}{2\sigma^2\theta}} dz. \]
Black-Scholes-Merton Approach

The explicit solution can be simplified as

\[ u(t, x) = x\Phi(g(t, x)) - Ke^{-rt}\Phi(h(t, x)), \]

where \( \Phi \) is the standard normal distribution function, and

\[ g(t, x) = \frac{\ln(x/K) + (r + 0.5\sigma^2)t}{\sigma t^{1/2}}, \quad h(t, x) = g(t, x) - \sigma t^{1/2}. \]
Black-Scholes-Merton Approach

The explicit solution can be simplified as

$$u(t, x) = x\Phi(g(t, x)) - Ke^{-rt}\Phi(h(t, x)),$$

where $\Phi$ is the standard normal distribution function, and

$$g(t, x) = \ln(x/K) + (r + 0.5\sigma^2)t$$
$$h(t, x) = g(t, x) - \sigma t^{1/2}.$$

Black-Scholes Option Pricing Formula

A rational price at time $t = 0$ for a European call option with exercise price $K$ is

$$V_0 = X_0\Phi(g(T, X_0)) - Ke^{-rT}\Phi(h(T, X_0)).$$
Black-Scholes-Merton Approach

The explicit solution can be simplified as

\[ u(t, x) = x\Phi(g(t, x)) - Ke^{-rt}\Phi(h(t, x)), \]

where \( \Phi \) is the standard normal distribution function, and

\[
g(t, x) = \frac{\ln(x/K) + (r + 0.5\sigma^2)t}{\sigma t^{1/2}}, \quad h(t, x) = g(t, x) - \sigma t^{1/2}.
\]

### Black-Scholes Option Pricing Formula

A rational price at time \( t = 0 \) for a European call option with exercise price \( K \) is

\[ V_0 = X_0\Phi(g(T, X_0)) - Ke^{-rT}\Phi(h(T, X_0)). \]

The stochastic process \( V_t = u(T - t, X_t) \) is the value of your self-financing portfolio with trading strategy

\[
a_t = u_2(T - t, X_t) > 0, \quad b_t = \frac{u(T - t, X_t) - a_tX_t}{\beta_t}.
\]
The Radon-Nikodym Theorem

Consider two measures $\mu$ and $\nu$ defined on a $\sigma$-field $\mathcal{F}$ on $\Omega$. $\mu$ is said to be absolutely continuous with respect to $\nu$ (denoted by $\mu \ll \nu$) if

$$\nu(A) = 0 \text{ implies } \mu(A) = 0, \quad \forall A \in \mathcal{F}.$$ 

We say that $\mu$ and $\nu$ are equivalent measures if $\mu \ll \nu$ and $\nu \ll \mu$. 

Theorem
Assume $\mu$ and $\nu$ are two $\sigma$-finite measures. Then $\mu \ll \nu$ holds if and only if there exists a non-negative measurable function $f$ such that

$$\mu(A) = \int_A f(\omega) \, d\nu(\omega), \quad \forall A \in \mathcal{F}.$$ 

Moreover, $f$ is almost everywhere unique with respect to $\nu$. The function $f$ is called the (relative) density of $\mu$ with respect to $\nu$, and denoted by $f = \frac{d\mu}{d\nu}$. 

Haijun Li
The Radon-Nikodym Theorem

Consider two measures $\mu$ and $\nu$ defined on a $\sigma$-field $\mathcal{F}$ on $\Omega$. $\mu$ is said to be absolutely continuous with respect to $\nu$ (denoted by $\mu \ll \nu$) if

$$\nu(A) = 0 \text{ implies } \mu(A) = 0, \forall A \in \mathcal{F}.$$ 

We say that $\mu$ and $\nu$ are equivalent measures if $\mu \ll \nu$ and $\nu \ll \mu$.

**Theorem**

Assume $\mu$ and $\nu$ are two $\sigma$-finite measures. Then $\mu \ll \nu$ holds if and only if there exists a non-negative measurable function $f$ such that

$$\mu(A) = \int_A f(\omega) d\nu(\omega), \forall A \in \mathcal{F}.$$ 

Moreover, $f$ is almost everywhere unique with respect to $\nu$. The function $f$ is called the (relative) density of $\mu$ with respect to $\nu$, and denoted by $f = \frac{d\mu}{d\nu}$. 
Girsanov’s Theorem

Let \( B = (B_t, t \geq 0) \) be standard Brownian motion on the probability space \((\Omega, \mathcal{F}, P)\), and \( \mathcal{F}_t = \sigma(B_s, s \leq t) \) the Brownian filtration. Consider

\[
\tilde{B}_t = B_t + qt, \quad t \in [0, T], \text{ for some constant } q.
\]

Although \( \tilde{B} \) is not a standard Brownian motion under \( P \) for \( q \neq 0 \), \( \tilde{B} \) can be shown to be a standard Brownian motion under the new probability measure \( Q \).
Girsanov’s Theorem

Let $B = (B_t, t \geq 0)$ be standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_t = \sigma(B_s, s \leq t)$ the Brownian filtration. Consider

$$\tilde{B}_t = B_t + qt, \ t \in [0, T], \text{ for some constant } q.$$  

Although $\tilde{B}$ is not a standard Brownian motion under $P$ for $q \neq 0$, $\tilde{B}$ can be shown to be a standard Brownian motion under the new probability measure $Q$.

Girsanov-Cameron-Martin Theorem

1. The stochastic process

$$M_t = \exp\{-qB_t - \frac{1}{2}q^2 t\}, \ t \in [0, T]$$

is a martingale with respect to the natural Brownian filtration under the probability measure $P$. 

Haijun Li  An Introduction to Stochastic Calculus  Week 11 12 / 20
Eliminating the Drift Term

Girsanov-Cameron-Martin Theorem

2. \( Q(A) = \int_A M_T(\omega) dP(\omega), \ A \in \mathcal{F}, \) defines a probability measure \( Q \) (called an equivalent martingale measure) on \( \mathcal{F} \) that is equivalent to \( P \).
Eliminating the Drift Term

Girsanov-Cameron-Martin Theorem

2. $Q(A) = \int_A M_T(\omega) dP(\omega)$, $A \in \mathcal{F}$, defines a probability measure $Q$ (called an equivalent martingale measure) on $\mathcal{F}$ that is equivalent to $P$.

3. Under the probability measure $Q$, the process $\tilde{B}$ is a standard Brownian motion.
Eliminating the Drift Term

Girsanov-Cameron-Martin Theorem

2. $Q(A) = \int_A M_T(\omega) dP(\omega)$, $A \in \mathcal{F}$, defines a probability measure $Q$ (called an equivalent martingale measure) on $\mathcal{F}$ that is equivalent to $P$.

3. Under the probability measure $Q$, the process $\tilde{B}$ is a standard Brownian motion.

4. The process $\tilde{B}$ is adapted to the filtration $\mathcal{F}_t$. 

Haijun Li
An Introduction to Stochastic Calculus
Week 11 13 / 20
Eliminating the Drift Term

Girsanov-Cameron-Martin Theorem

2. \( Q(A) = \int_A M_T(\omega) dP(\omega) \), \( A \in \mathcal{F} \), defines a probability measure \( Q \) (called an equivalent martingale measure) on \( \mathcal{F} \) that is equivalent to \( P \).

3. Under the probability measure \( Q \), the process \( \tilde{B} \) is a standard Brownian motion.

4. The process \( \tilde{B} \) is adapted to the filtration \( \mathcal{F}_t \).

Consider the linear stochastic differential equation

\[
dX_t = cX_t dt + \sigma X_t dB_t, \quad t \in [0, T].
\]

With a linear drift term, \( X \) is not a martingale under \( P \).
Eliminating the Drift Term

Girsanov-Cameron-Martin Theorem

2. \( Q(A) = \int_A M_T(\omega) dP(\omega), A \in \mathcal{F}, \) defines a probability measure \( Q \) (called an equivalent martingale measure) on \( \mathcal{F} \) that is equivalent to \( P \).

3. Under the probability measure \( Q \), the process \( \tilde{B} \) is a standard Brownian motion.

4. The process \( \tilde{B} \) is adapted to the filtration \( \mathcal{F}_t \).

- Consider the linear stochastic differential equation

\[
    dX_t = cX_t dt + \sigma X_t dB_t, \quad t \in [0, T].
\]

With a linear drift term, \( X \) is not a martingale under \( P \).

- Define \( \tilde{B}_t = B_t + \frac{c}{\sigma} t \), and we have

\[
    dX_t = \sigma X_t d(B_t + \frac{c}{\sigma} t) = \sigma X_t d\tilde{B}_t, \quad t \in [0, T].
\]
Eliminating the Drift Term

Girsanov-Cameron-Martin Theorem

2. \[ Q(A) = \int_A M_T(\omega) dP(\omega), \ A \in \mathcal{F}, \] defines a probability measure \( Q \) (called an equivalent martingale measure) on \( \mathcal{F} \) that is equivalent to \( P \).

3. Under the probability measure \( Q \), the process \( \tilde{B} \) is a standard Brownian motion.

4. The process \( \tilde{B} \) is adapted to the filtration \( \mathcal{F}_t \).

- Consider the linear stochastic differential equation

\[ dX_t = cX_t dt + \sigma X_t dB_t, \ t \in [0, T]. \]

With a linear drift term, \( X \) is not a martingale under \( P \).

- Define \( \tilde{B}_t = B_t + \frac{c}{\sigma} t \), and we have

\[ dX_t = \sigma X_t d(B_t + \frac{c}{\sigma} t) = \sigma X_t d\tilde{B}_t, \ t \in [0, T]. \]

- \( \tilde{B} \) is a standard Brownian motion under the equivalent martingale measure \( Q \), and thus \( X \) is a martingale under \( Q \).
If we had known the solution only for the case without a linear drift, we could have derived the solution for the case with a linear drift via the change of measure.
If we had known the solution only for the case without a linear drift, we could have derived the solution for the case with a linear drift via the change of measure.

More significantly, $X$ is a martingale under the equivalent martingale measure $Q$, and one can make use of the martingale property for proving various results about $X$. 
Significance of the Change-of-Measure Trick

- If we had known the solution only for the case without a linear drift, we could have derived the solution for the case with a linear drift via the change of measure.
- More significantly, $X$ is a martingale under the equivalent martingale measure $Q$, and one can make use of the martingale property for proving various results about $X$.
- In fact, this is not just a technical trick, and as we demonstrate below, the change of measure provides an effective method to incorporate uncertainty and to hedge against contingent claims.
The price of one share of the risky asset (stock) is described by

\[ dX_t = cX_t dt + \sigma X_t dB_t, \quad t \in [0, T]. \]
Recap: The Black-Scholes Model

- The price of one share of the risky asset (stock) is described by

\[ dX_t = cX_t \, dt + \sigma X_t \, dB_t, \quad t \in [0, T]. \]

- The price of the riskless asset (bond) is described by

\[ d\beta_t = r \beta \, d\beta_t, \quad t \in [0, T]. \]
Recap: The Black-Scholes Model

- The price of one share of the risky asset (stock) is described by
  \[ dX_t = cX_t dt + \sigma X_t dB_t, \ t \in [0, T]. \]

- The price of the riskless asset (bond) is described by
  \[ d\beta_t = r\beta d\beta_t, \ t \in [0, T]. \]

- Portfolio = \((a_t, b_t)\), with value \( V_t = a_t X_t + b_t \beta_t \) at time \( t \).
Recap: The Black-Scholes Model

- The price of one share of the risky asset (stock) is described by
  \[ dX_t = cX_t dt + \sigma X_t dB_t, \ t \in [0, T]. \]

- The price of the riskless asset (bond) is described by
  \[ d\beta_t = r \beta d\beta_t, \ t \in [0, T]. \]

- Portfolio = \((a_t, b_t)\), with value \( V_t = a_t X_t + b_t \beta_t \) at time \( t \).

- The portfolio is self-financing: \( dV_t = a_t dX_t + b_t d\beta_t, \ t \in [0, T]. \)
The price of one share of the risky asset (stock) is described by

\[ dX_t = cX_t dt + \sigma X_t dB_t, \quad t \in [0, T]. \]

The price of the riskless asset (bond) is described by

\[ d\beta_t = r \beta_t dB_t, \quad t \in [0, T]. \]

Portfolio = \((a_t, b_t)\), with value \(V_t = a_t X_t + b_t \beta_t\) at time \(t\).

The portfolio is self-financing: \(dV_t = a_t dX_t + b_t d\beta_t, \quad t \in [0, T]\).

At time of maturity, \(V_T = h(X_T)\), where \(h(X_t)\) is the contingent claim at time \(t\). For a European call option, \(h(x) = (x - K)^+\), and for a European put option, \(h(x) = (K - x)^+\).
Your gain from the option at time of maturity is $h(X_T)$. To determine the value of this amount of money at $t = 0$, you have to discount it with given interest rate $r$: $e^{-rT}h(X_T)$, and take the expectation of it as the price for the option at $t = 0$. 
Your gain from the option at time of maturity is $h(X_T)$. To determine the value of this amount of money at $t = 0$, you have to discount it with given interest rate $r$: $e^{-rT}h(X_T)$, and take the expectation of it as the price for the option at $t = 0$.

You have to also discount price of one share of stock: $	ilde{X}_t = e^{-rt}X_t$, $t \in [0, T]$, and the Itô lemma leads to $d\tilde{X}_t = \sigma \tilde{X}_td\tilde{B}_t$, where $\tilde{B}_t = B_t + \frac{c-r}{\sigma}t$. 
Pricing via the Change-of-Measure

- Your gain from the option at time of maturity is \( h(X_T) \). To determine the value of this amount of money at \( t = 0 \), you have to discount it with given interest rate \( r \): \( e^{-rT} h(X_T) \), and take the expectation of it as the price for the option at \( t = 0 \).
- You have to also discount price of one share of stock: \( \tilde{X}_t = e^{-rt} X_t, t \in [0, T] \), and the Itô lemma leads to \( d\tilde{X}_t = \sigma \tilde{X}_t d\tilde{B}_t \), where \( \tilde{B}_t = B_t + \frac{c-r}{\sigma} t \).
- There exists an equivalent martingale measure \( Q \) which turns \( \tilde{B} \) into a standard Brownian motion, and

\[
\tilde{X}_t = \tilde{X}_0 e^{-0.5 \sigma^2 t + \sigma \tilde{B}_t}
\]

becomes a martingale with respect to the natural Brownian filtration under \( Q \).
Your gain from the option at time of maturity is $h(X_T)$. To determine the value of this amount of money at $t = 0$, you have to discount it with given interest rate $r$: $e^{-rT}h(X_T)$, and take the expectation of it as the price for the option at $t = 0$.

You have to also discount price of one share of stock: $	ilde{X}_t = e^{-rt}X_t$, $t \in [0, T]$, and the Itô lemma leads to $d\tilde{X}_t = \sigma \tilde{X}_t d\tilde{B}_t$, where $\tilde{B}_t = B_t + \frac{c-r}{\sigma}t$.

There exists an equivalent martingale measure $Q$ which turns $\tilde{B}$ into a standard Brownian motion, and

$$\tilde{X}_t = \tilde{X}_0 e^{-0.5\sigma^2 t + \sigma \tilde{B}_t}$$

becomes a martingale with respect to the natural Brownian filtration under $Q$.

The value of the portfolio at time $t$ is given by

$$V_t = E_Q[e^{-r(T-t)}h(X_T)|\mathcal{F}_t], \ t \in [0, T].$$
Pricing via the Change-of-Measure

- Your gain from the option at time of maturity is $h(X_T)$. To determine the value of this amount of money at $t = 0$, you have to discount it with given interest rate $r$: $e^{-rT}h(X_T)$, and take the expectation of it as the price for the option at $t = 0$.

- You have to also discount price of one share of stock: $\tilde{X}_t = e^{-rt}X_t$, $t \in [0, T]$, and the Itô lemma leads to $d\tilde{X}_t = \sigma \tilde{X}_t d\tilde{B}_t$, where $\tilde{B}_t = B_t + \frac{c-r}{\sigma} t$.

- There exists an equivalent martingale measure $Q$ which turns $\tilde{B}$ into a standard Brownian motion, and

$$\tilde{X}_t = \tilde{X}_0 e^{-0.5\sigma^2 t + \sigma \tilde{B}_t}$$

becomes a martingale with respect to the natural Brownian filtration under $Q$.

- The value of the portfolio at time $t$ is given by

$$V_t = E_Q[e^{-r(T-t)}h(X_T)|F_t], \quad t \in [0, T]$$

- At time $t = 0$, $V_0 = E_Q[e^{-rT}h(X_T)]$ is a rational price of the option.
The Value of a European Option

- Write $\theta = T - t$ for $t \in [0, T]$. 
The Value of a European Option

- Write \( \theta = T - t \) for \( t \in [0, T] \).

- Since \( X_t = X_0 e^{(r - 0.5 \sigma^2) t + \sigma \tilde{B}_t} \), we have
  \[
  V_t = E_Q \left[ e^{-r \theta} h(X_t e^{(r - 0.5 \sigma^2) \theta + \sigma (\tilde{B}_T - \tilde{B}_t)}) | \mathcal{F}_t \right].
  \]
Write \( \theta = T - t \) for \( t \in [0, T] \).

Since \( X_t = X_0 e^{(r-0.5\sigma^2)t+\sigma \tilde{B}_t} \), we have

\[
V_t = E_Q\left[ e^{-r\theta} h(X_t e^{(r-0.5\sigma^2)\theta+\sigma(\tilde{B}_T-\tilde{B}_t)}) | \mathcal{F}_t \right].
\]

Since \( \sigma(X_t) \subseteq \mathcal{F}_t \), \( X_t \) can be treated as a constant under \( \mathcal{F}_t \).
The Value of a European Option

- Write \( \theta = T - t \) for \( t \in [0, T] \).
- Since \( X_t = X_0 e^{(r-0.5\sigma^2)t+\sigma\tilde{B}_t} \), we have
  \[ V_t = E_Q\left[ e^{-r\theta} h(X_t e^{(r-0.5\sigma^2)\theta+\sigma(\tilde{B}_T-\tilde{B}_t)}) | \mathcal{F}_t \right]. \]
- Since \( \sigma(X_t) \subseteq \mathcal{F}_t \), \( X_t \) can be treated as a constant under \( \mathcal{F}_t \).
- Under \( Q \), \( \tilde{B}_T - \tilde{B}_t \sim N(0, \theta) \), and is independent of \( \mathcal{F}_t \).
The Value of a European Option

- Write $\theta = T - t$ for $t \in [0, T]$.
- Since $X_t = X_0 e^{(r-0.5\sigma^2)t + \sigma \tilde{B}_t}$, we have
  $V_t = E_Q \left[ e^{-r\theta} h(X_t e^{(r-0.5\sigma^2)\theta + \sigma (\tilde{B}_T - \tilde{B}_t)}) | F_t \right]$. 
- Since $\sigma(X_t) \subseteq F_t$, $X_t$ can be treated as a constant under $F_t$.
- Under $Q$, $\tilde{B}_T - \tilde{B}_t \sim N(0, \theta)$, and is independent of $F_t$.
- Thus, $V_t = f(t, X_t)$, where
  \[ f(t, x) = e^{-r\theta} \int_{-\infty}^{\infty} h(x e^{(r-0.5\sigma^2)\theta + \sigma y \theta^{1/2}}) d\Phi(y). \]
Write \( \theta = T - t \) for \( t \in [0, T] \).

Since \( X_t = X_0 e^{(r-0.5\sigma^2)t + \sigma \tilde{B}_t} \), we have
\[
V_t = E_Q \left[ e^{-r\theta} h(X_t e^{(r-0.5\sigma^2)\theta + \sigma (\tilde{B}_T - \tilde{B}_t)}) \mid \mathcal{F}_t \right].
\]

Since \( \sigma(X_t) \subseteq \mathcal{F}_t \), \( X_t \) can be treated as a constant under \( \mathcal{F}_t \).

Under \( Q \), \( \tilde{B}_T - \tilde{B}_t \sim N(0, \theta) \), and is independent of \( \mathcal{F}_t \).

Thus, \( V_t = f(t, X_t) \), where
\[
f(t, x) = e^{-r\theta} \int_{-\infty}^{\infty} h(x e^{(r-0.5\sigma^2)\theta + \sigma y \theta^{1/2}}) d\Phi(y).
\]

For a European call option, \( h(x) = (x - K)^+ \), and thus
\[
f(t, x) = x \Phi(z_1) - Ke^{-r\theta} \Phi(z_2),
\]
where
\[
z_1 = \frac{\ln(x/K) + (r + 0.5\sigma^2)\theta}{\sigma \theta^{1/2}}, \quad z_2 = z_1 - \sigma \theta^{1/2}.
\]
The Value of a European Option

- Write $\theta = T - t$ for $t \in [0, T]$.
- Since $X_t = X_0 e^{(r-0.5\sigma^2)t+\sigma \tilde{B}_t}$, we have
  $$V_t = E_Q \left[ e^{-r\theta} h(X_t e^{(r-0.5\sigma^2)\theta+\sigma(\tilde{B}_T-\tilde{B}_t)}) | \mathcal{F}_t \right].$$
- Since $\sigma(X_t) \subseteq \mathcal{F}_t$, $X_t$ can be treated as a constant under $\mathcal{F}_t$.
- Under $Q$, $\tilde{B}_T - \tilde{B}_t \sim N(0, \theta)$, and is independent of $\mathcal{F}_t$.
- Thus, $V_t = f(t, X_t)$, where
  $$f(t, x) = e^{-r\theta} \int_{-\infty}^{\infty} h(xe^{(r-0.5\sigma^2)\theta+\sigma y^{1/2}}) d\Phi(y).$$
- For a European call option, $h(x) = (x - K)^+$, and thus
  $$f(t, x) = x\Phi(z_1) - Ke^{-r\theta}\Phi(z_2),$$
  $$z_1 = \frac{\ln(x/K) + (r + 0.5\sigma^2)\theta}{\sigma\theta^{1/2}}, \quad z_2 = z_1 - \sigma\theta^{1/2}.$$  
- For a European put option, $f(t, x) = Ke^{-r\theta}\Phi(-z_2) - x\Phi(-z_1)$. 

Haijun Li
An Introduction to Stochastic Calculus
Week 11 17 / 20
The Black-Scholes model can be extended for variable (but deterministic) rates and volatilities.
Extensions and Limitations of the Model

- The Black-Scholes model can be extended for variable (but deterministic) rates and volatilities.
- The model may be also used to value European style options on instruments paying dividends, and closed-form solutions available if the dividend is a known proportion of the stock price.
Extensions and Limitations of the Model

- The Black-Scholes model can be extended for variable (but deterministic) rates and volatilities.
- The model may be also used to value European style options on instruments paying dividends, and closed-form solutions available if the dividend is a known proportion of the stock price.
- The model underestimates extreme moves that yields tail risk.
Extensions and Limitations of the Model

- The Black-Scholes model can be extended for variable (but deterministic) rates and volatilities.
- The model may be also used to value European style options on instruments paying dividends, and closed-form solutions available if the dividend is a known proportion of the stock price.
- The model underestimates extreme moves that yields tail risk.
- In reality security prices do not follow a strict stationary log-normal process, nor is the risk-free interest actually known (and is not constant over time).
Extensions and Limitations of the Model

- The Black-Scholes model can be extended for variable (but deterministic) rates and volatilities.
- The model may be also used to value European style options on instruments paying dividends, and closed-form solutions available if the dividend is a known proportion of the stock price.
- The model underestimates extreme moves that yields tail risk.
- In reality security prices do not follow a strict stationary log-normal process, nor is the risk-free interest actually known (and is not constant over time).
- The variance has been observed to be non-constant leading to models such as GARCH to model volatility changes.
Extensions and Limitations of the Model

- The Black-Scholes model can be extended for variable (but deterministic) rates and volatilities.
- The model may be also used to value European style options on instruments paying dividends, and closed-form solutions available if the dividend is a known proportion of the stock price.
- The model underestimates extreme moves that yields tail risk.
- In reality security prices do not follow a strict stationary log-normal process, nor is the risk-free interest actually known (and is not constant over time).
- The variance has been observed to be non-constant leading to models such as GARCH to model volatility changes.
- Pricing discrepancies between empirical and the Black-Scholes model have long been observed in options corresponding to extreme price changes; such events would be very rare if returns were log-normally distributed, but are observed much more often in practice.
A sociologist investigating the behavior of the probability community during the early 1990s would surely report an interesting phenomenon. Many of the best minds of this (or any other) generation began concentrating their research in the area of mathematical finance. The main reason for this can be summed up in two words: option pricing (D. Applebaum, 2004)
A sociologist investigating the behavior of the probability community during the early 1990s would surely report an interesting phenomenon. Many of the best minds of this (or any other) generation began concentrating their research in the area of mathematical finance. The main reason for this can be summed up in two words: option pricing (D. Applebaum, 2004)

The Black-Scholes model is widely employed as a useful approximation, but proper application requires understanding its limitations.
A sociologist investigating the behavior of the probability community during the early 1990s would surely report an interesting phenomenon. Many of the best minds of this (or any other) generation began concentrating their research in the area of mathematical finance. The main reason for this can be summed up in two words: option pricing (D. Applebaum, 2004).

The Black-Scholes model is widely employed as a useful approximation, but proper application requires understanding its limitations.

The limitations and defects of the model have led many probabilists to query it.
Lévy Matters

- Heavy tails of stock prices, which is incompatible with a Gaussian model, suggests that it might be fruitful to replace Brownian motion with a more general Lévy process.
Lévy Matters

- Heavy tails of stock prices, which is incompatible with a Gaussian model, suggests that it might be fruitful to replace Brownian motion with a more general Lévy process.
- A Lévy process $L = (L_t, t \geq 0)$ has independent and stationary increments and is stochastically continuous, i.e.,
  \[
  \lim_{t \to s} P(|L_t - L_s| > \epsilon) = 0 \text{ for any } \epsilon > 0.
  \]
Lévy Matters

- Heavy tails of stock prices, which is incompatible with a Gaussian model, suggests that it might be fruitful to replace Brownian motion with a more general Lévy process.

- A Lévy process \( L = (L_t, t \geq 0) \) has independent and stationary increments and is stochastically continuous, i.e.,
  \[
  \lim_{t \to s} P(|L_t - L_s| > \epsilon) = 0 \quad \text{for any } \epsilon > 0.
  \]

- Example: Brownian motion, the Poisson process, compound Poisson processes and their “combinations”.

The Lévy-Itô decomposition for a one-dimensional Lévy process:

\[
L_t = b_t + B_t + \int_{|x| < 1} x \left( N(t, dx) - t \nu(dx) \right) + \int_{|x| \geq 1} x N(t, dx),
\]

where \( N = \text{Poisson random measure} \) and \( \nu = \text{the Lévy measure} \).

The small jumps term \( \int_{|x| < 1} x \left( N(t, dx) - t \nu(dx) \right) \) describes the day-to-day jitter that causes minor fluctuations in stock prices, while the big jumps term \( \int_{|x| \geq 1} x N(t, dx) \) describes large stock price movements caused by major market upsets arising from, e.g., earthquakes or terrorist atrocities.
Heavy tails of stock prices, which is incompatible with a Gaussian model, suggests that it might be fruitful to replace Brownian motion with a more general Lévy process.

A Lévy process \( L = (L_t, t \geq 0) \) has independent and stationary increments and is stochastically continuous, i.e.,
\[
\lim_{t \to s} P(|L_t - L_s| > \epsilon) = 0 \quad \text{for any } \epsilon > 0.
\]

Example: Brownian motion, the Poisson process, compound Poisson processes and their “combinations”.

The Lévy-Itô decomposition for a one-dimensional Lévy process:
\[
L_t = bt + B_t + \int_{|x|<1} x(N(t, dx) - t \nu(dx)) + \int_{|x|\geq1} xN(t, dx),
\]
where \( N = \) Poisson random measure and \( \nu = \) the Lévy measure.
Lévy Matters

- Heavy tails of stock prices, which is incompatible with a Gaussian model, suggests that it might be fruitful to replace Brownian motion with a more general Lévy process.
- A Lévy process $L = (L_t, t \geq 0)$ has independent and stationary increments and is stochastically continuous, i.e.,
  $\lim_{t \to s} P(|L_t - L_s| > \epsilon) = 0$ for any $\epsilon > 0$.
- Example: Brownian motion, the Poisson process, compound Poisson processes and their “combinations”.
- The Lévy-Itô decomposition for a one-dimensional Lévy process:
  
  $$L_t = bt + B_t + \int_{|x|<1} x(N(t, dx) - t\nu(dx)) + \int_{|x|\geq1} xN(t, dx),$$

  where $N$ = Poisson random measure and $\nu = $ the Lévy measure.
- The small jumps term $\int_{|x|<1} x(N(t, dx) - t\nu(dx))$ describes the day-to-day jitter that causes minor fluctuations in stock prices, while the big jumps term $\int_{|x|\geq1} xN(t, dx)$ describes large stock price movements caused by major market upsets arising from, e.g., earthquakes or terrorist atrocities.