Outline

1. Coefficients of Tail Dependence

2. Simulation and Calibration of Copulas

3. Parameter Estimation and Calibration of Copulas
Tail Dependence = Dependence in the Tails

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**Definition (Joe, 1997)**

Let $X_1$ and $X_2$ be two random variables with continuous marginal distributions $F_1$ and $F_2$ respectively.

1. The coefficient of lower tail dependence of $X, Y$ is defined as

$$\lambda_l := \lim_{u \to 0} P(X_1 \leq F_1^{-1}(u)|X_2 \leq F_2^{-1}(u)),$$

provided that the limit exists.

2. The coefficient of upper tail dependence of $X, Y$ is defined as

$$\lambda_u := \lim_{u \to 1} P(X_1 > F_1^{-1}(u)|X_2 > F_2^{-1}(u)),$$

provided that the limit exists.

Note that $F_1^{-1}(u)$ and $F_2^{-1}(u)$ are 100$u$% percentiles, $0 \leq u \leq 1$. 
Properties of Tail Dependence Coefficients

Let $X_1$ and $X_2$ be two random variables with copula $C$.

The tail dependence coefficients are rank-invariant (i.e., invariant under increasing marginal transforms), because

$$
\lambda_l = \lim_{u \to 0} \frac{C(u, u)}{u}, \quad \text{and} \quad \lambda_u = \lim_{u \to 0} \frac{2u - 1 + C(1 - u, 1 - u)}{u}.
$$

Let

$$
\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2), \quad (u_1, u_2) \in [0, 1]^2.
$$

This is called the survival copula of $X_1, X_2$.

For any (elliptically) symmetric distribution (such as bivariate normal and $t$ distributions),

$$
C(u_1, u_2) = \hat{C}(u_1, u_2), \quad (u_1, u_2) \in [0, 1]^2,
$$

which implies that $\lambda_l = \lambda_u$. 
Example: Clayton Copulas

- The Laplace transform $\varphi(t) = (t + 1)^{-1/\theta}$, $t \geq 0$, with inverse $\varphi^{-1}(u) = u^{-\theta} - 1, 0 \leq u \leq 1$.

- Consider a bivariate Clayton copula:

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}, \forall (u_1, u_2) \in [0, 1]^2.$$

- Verify that $\lambda_u = 0$.

- Verify that $\lambda_l = 2^{-1/\theta}$, where $\theta > 0$. 
Example (cont’d): Cayton VS Gaussian

Figure: Left: Kendall’s $\tau = 0.71$ Right: Kendall’s $\tau = 0.67$
Example: Gumbel Copulas

- The inverse $\varphi^{-1}(u) = (-\ln u)^\theta$, $0 \leq u \leq 1$.
- Consider a bivariate Gumbel copula:
  \[
  C(u; \theta) = \exp \left\{ - \left[ (-\ln u_1)^\theta + (-\ln u_2)^\theta \right]^{1/\theta} \right\}, \ \forall \ (u_1, u_2) \in [0, 1]^2.
  \]
- Verify that $\lambda_l = 0$.
- Verify that $\lambda_u = 2 - 2^{1/\theta}$, where $\theta > 1$. 
Example: Frank Copulas

- The inverse $\varphi^{-1}(u) = -\ln[(e^{-\theta u} - 1)/(e^{-\theta} - 1)], \ 0 \leq u \leq 1$.
- In the bivariate case, this yields the Frank copula:

$$C(u_1, u_2; \theta) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{(e^{-\theta} - 1)}\right).$$

- Verify that $\lambda_l = 0$.
- Verify that $\lambda_u = 0$. 
Total Probability Laws in the Tails

Let $X_1$ and $X_2$ be two random variables with copula $C$ and marginal distributions $F_1$ and $F_2$ respectively. Let $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$.

**Theorem**

\[
\lambda_l = \lim_{u \to 0} \left[ P(U_2 \leq u \mid U_1 = u) + P(U_1 \leq u \mid U_2 = u) \right] = \lim_{u \to 0} \frac{dC(u, u)}{du}.
\]

\[
\lambda_u = \lim_{u \to 0} \left[ P(U_2 > 1 - u \mid U_1 = 1 - u) + P(U_1 > 1 - u \mid U_2 = 1 - u) \right] = \lim_{u \to 0} \frac{\hat{dC}(u, u)}{du}.
\]
Example: Gaussian Copula

Let \((U_1, U_2)\) has a Gaussian copula \(C_\rho\) with correlation coefficient \(\rho\).

That is, \((X_1, X_2) = (\Phi^{-1}(U_1), \Phi^{-1}(U_2))\) has a bivariate Gaussian distribution with correlation coefficient \(\rho\) and standard normal marginal distributions.

If \(\rho < 1\), then \(\lambda_l = \lambda_u = 0\).
Example: t Copula

- Let \((U_1, U_2)\) has a t copula \(C_{\nu,\rho}\) with correlation coefficient \(\rho\) and d.f. \(\nu\).
- Let \(F_{\nu}\) denote the cumulative function of the t distribution with \(\nu\) degrees of freedom.
- Then \((X_1, X_2) = (F_{\nu}^{-1}(U_1), F_{\nu}^{-1}(U_2))\) has a bivariate t distribution with correlation coefficient \(\rho\) and standard t marginal distributions.
- The tail dependence coefficient: If \(\rho > -1\), then

\[
\lambda_l = \lambda_u = 2F_{\nu+1}\left(-\sqrt{\frac{(\nu + 1)(1 - \rho)}{1 + \rho}}\right).
\]
Example: Clayton Copulas

Consider a bivariate Clayton copula:

$$C(u_1, u_2; \theta) = \left( u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-1/\theta}, \quad \forall (u_1, u_2) \in [0, 1]^2.$$ 

Since

$$C_{2|1}(u_2 | u_1) = \frac{\partial C(u_1, u_2)}{\partial u_1} = (1 + u_1^\theta u_2^{-\theta} - u_1^\theta)^{-1-1/\theta},$$

then

$$C_{2|1}^{-1}(u_2 | u_1) = u_1 (u_2^{-\theta/(1+\theta)} + u_1^\theta - 1)^{-1/\theta}.$$ 

**Algorithm:**

1. Generate two independent samples $\nu_1, \nu_2$ from the standard uniform distribution over $[0, 1]$;
2. $u_1 := \nu_1$;
3. $u_2 := u_1 (\nu_2^{-\theta/(1+\theta)} + u_1^\theta - 1)^{-1/\theta}$;
4. return $(u_1, u_2)$. 
Let \((U_1, \ldots, U_d) \sim C\).

The conditional distributions are given as follows, for \(1 \leq j \leq d - 1\),

\[
C_{j+1|1,...,j}(u_{j+1}|u_1, \ldots, u_j) = P(U_{j+1} \leq u_{j+1} \mid U_1 = u_1, \ldots, U_j = u_j) = \frac{\partial^j C(u_1, \ldots, u_j, u_{j+1}, 1, \ldots, 1)}{\partial u_1, \ldots, \partial u_j} / \frac{\partial^j C(u_1, \ldots, u_j, 1, \ldots, 1)}{\partial u_1, \ldots, \partial u_j}.
\]

### Conditioning Sampling Algorithm:

1. Generate \(d\) independent random variates \(v_1, \ldots, v_d\) from the standard uniform distribution on \([0, 1]\).
   - Set \(u_1 := v_1\).
   - Given \(u_1, \ldots, u_j\), set \(u_{j+1} := C_{j+1|1,...,j}^{-1}(v_{j+1}|u_1, \ldots, u_j)\), for \(j = 1, \ldots, n - 1\).

2. \((u_1, \ldots, u_d)\) is a sample generated from \(C\).
Consider the following stochastic representation for the $d$-dimensional t-distribution with $\nu > 1$ degrees of freedom and dispersion matrix $\Sigma = (\sigma_{ij})$:

$$(X_1, \ldots, X_d) = \frac{\sqrt{\nu}}{\sqrt{S}} Z,$$

where $S \sim \chi^2_\nu$ and $Z = (Z_1, \ldots, Z_d) \sim N(0, \Sigma)$ are independent.

Let $F_\nu(x)$ denote the standard t distribution with d.f. $\nu$. Then $(F_\nu(X_1), \ldots, F_\nu(X_d))$ has the t copula.
Sampling from a t-Copula

Let $F_\nu$ denote the $i$-th marginal df of $X_i$, $1 \leq i \leq d$, and $A$ denote the Cholesky decomposition of matrix $\Sigma = (\sigma_{ij})$; that is, $A^\top A = \Sigma$.

Algorithm:

1. Generate $d$ independent variates $\mathbf{z} = (z_1, \ldots, z_d)$ from $\mathcal{N}(0, 1)$.
2. Generate a variate $s$ from $\chi^2_\nu$ that are independent of $z_1, \ldots, z_d$.
3. Set $(x_1, \ldots, x_d) = (\sqrt{\nu} s^{-1/2} \mathbf{z} A)$.
4. Set $u_i = F_\nu(x_i), 1 \leq i \leq d$.
5. $(u_1, \ldots, u_d)$ is a sample generated from the t copula $C$ with $\nu > 1$ degrees of freedom and dispersion matrix $\Sigma$. 
Given data: $n$ observations $(x_{1j}, \ldots, x_{dj}), j = 1, \ldots, n$.

Goal: Calibrating a copula $C$ with proper marginal distribution $F_1, \ldots, F_d$. 
Parameter Estimation and Calibration of Copulas

**Given data:** $n$ observations $(x_{1j}, \ldots, x_{dj}), j = 1, \ldots, n$.

**Goal:** Calibrating a copula $C$ with proper marginal distribution $F_1, \ldots, F_d$.

**Parametric Model**

Consider a copula-based parametric model for the $d$-dimensional random vector $\mathbf{x}$ with df

$$F(\mathbf{x}; \alpha_1, \ldots, \alpha_d, \theta) = C(F_1(x_1; \alpha_1), \ldots, F_d(x_d; \alpha_d); \theta), \quad \mathbf{x} = (x_1, \ldots, x_d),$$

where $F_i, 1 \leq i \leq d$, is the $i$-th univariate marginal df with density $f_i$ parametrized by a vector $\alpha_i$ of parameters, and $C$ is a copula with density $c$ parametrized by a vector $\theta$ of parameters.
Two-Step Inference

Given data: \( n \) observations \((x_{1j}, \ldots, x_{dj}), j = 1, \ldots, n\).

The log-likelihood function is given by

\[
L(\alpha_1, \ldots, \alpha_d, \theta) = \sum_{i=1}^{d} \sum_{j=1}^{n} \log f_i(x_{ij}; \alpha_i) + \sum_{j=1}^{n} \log c(F_1(x_{1j}; \alpha_1), \ldots, F_d(x_{dj}; \alpha_d); \theta).
\]

Algorithm:

1. The log-likelihoods \( L_i(\alpha_i) = \sum_{j=1}^{n} \log f_i(x_{ij}; \alpha_i) \) of the \( d \) univariate margins are separately maximized to obtain estimates \( \hat{\alpha}_1, \ldots, \hat{\alpha}_d \).
2. The function \( L(\hat{\alpha}_1, \ldots, \hat{\alpha}_d, \theta) \) is maximized over \( \theta \) to get \( \hat{\theta} \).
Remarks

- The vector of two-step estimators \((\hat{\alpha}_1, \ldots, \hat{\alpha}_d, \hat{\theta})\) is asymptotically normal distributed.

- In step 1, the marginal distributions \(F_1, \ldots, F_d\) can be fitted using empirical distribution functions:

\[
\hat{F}_i(x) := \frac{\sum_{j=1}^{n} I\{x_{ij} \leq x\}}{n} = \frac{\text{# of } x_{ij}'s \text{ that are less or equal to } x}{n}.
\]

Note that \(\hat{F}_i(x)\) is unbiased and a consistent estimator of \(F_i(x)\).