Pickands Representation
Multivariate Extremes

- \( X_n = (X_{1,n}, \ldots, X_{d,n}), \ n = 1, 2, \ldots, \) are iid with a \( d \)-dimensional distribution \( F(x_1, \ldots, x_d) \) with marginal distributions \( F_1(x_1), \ldots, F_d(x_d) \):
  
  \[ F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \quad (x_1, \ldots, x_d) \in \mathbb{R}^d_+. \]

- \( M_n := (M_{1,n}, \ldots, M_{d,n}) \), where the \( i \)-th component maximum
  
  \[ M_{i,n} := \max\{X_{i,1}, \ldots, X_{i,n}\} =: \bigvee_{j=1}^n X_{i,j}, \quad 1 \leq i \leq d. \]

- The limiting distribution, if exists, of
  
  \[ \frac{M_n - c_n}{d_n} = \left( \frac{M_{1,n} - c_{1,n}}{d_{1,n}}, \ldots, \frac{M_{d,n} - c_{d,n}}{d_{d,n}} \right) \]

  is denoted by \( H(x_1, \ldots, x_d) \) with marginal distributions \( H_1(x_1), \ldots, H_d(x_d) \):
  
  \[ H(x_1, \ldots, x_d) = C_{EV}(H_1(x_1), \ldots, H_d(x_d)), \quad (x_1, \ldots, x_d) \in \mathbb{R}^d_+. \]
Theorem

The following two statements are equivalent:

1. \( F \in \text{MDA}(H) \); that is,

\[
\lim_{n \to \infty} P \left( \frac{M_n - c_n}{d_n} \leq x \right) = \lim_{n \to \infty} F^n(c_n + d_n x) = H(x), \ x \in \mathbb{R}^d,
\]

for all continuity points \( x \) of \( H \).

2. \( F_i \in \text{MDA}(H_i), \ 1 \leq i \leq d \), and

\[
C_{EV}(u_1^t, \ldots, u_d^t) = C_{EV}^t(u_1, \ldots, u_d), \ (u_1, \ldots, u_d) \in [0, 1]^d.
\]
Multivariate Maximum Domain of Attraction

**Theorem**

The following two statements are equivalent:

1. $F \in \text{MDA}(H)$; that is,
   \[
   \lim_{n \to \infty} P \left( \frac{M_n - c_n}{d_n} \leq x \right) = \lim_{n \to \infty} F^n(c_n + d_n x) = H(x), \quad x \in \mathbb{R}^d,
   \]
   for all continuity points $x$ of $H$.

2. $F_i \in \text{MDA}(H_i), \quad 1 \leq i \leq d$, and
   \[
   C_{\text{EV}}(u_1^t, \ldots, u_d^t) = C_{\text{EV}}^t(u_1, \ldots, u_d), \quad (u_1, \ldots, u_d) \in [0, 1]^d.
   \]

- What is the relation between $C(u_1, \ldots, u_d)$ and $C_{\text{EV}}(u_1, \ldots, u_d)$?
- Can we obtain a better representation for $C_{\text{EV}}(u_1, \ldots, u_d)$?
Copula Domain of Attraction

Recall:

\[ F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \quad (x_1, \ldots, x_d) \in \mathbb{R}_+^d. \]

\[ H(x_1, \ldots, x_d) = C_{EV}(H_1(x_1), \ldots, H_d(x_d)), \quad (x_1, \ldots, x_d) \in \mathbb{R}_+^d. \]

**Theorem (Janos Galambos, 1975)**

\( F \in \text{MDA}(H) \) if and only if

1. \( F_i \in \text{MDA}(H_i), \ 1 \leq i \leq d; \) and
2. for any \((u_1, \ldots, u_d) \in [0, 1]^d\),

\[ \lim_{n \to \infty} C^n(u_1^{1/n}, \ldots, u_d^{1/n}) = C_{EV}(u_1, \ldots, u_d). \]

The limit in (2) is known as the copula domain of attraction.
Exponent Function

Since \( -\log x \approx 1 - x \) as \( x \uparrow 1 \), we have

\[
\lim_{n \to \infty} C^n(u_1^{1/n}, \ldots, u_d^{1/n}) = C_{EV}(u_1, \ldots, u_d)
\]

\[
\iff \quad \lim_{n \to \infty} n(1 - C(u_1^{1/n}, \ldots, u_d^{1/n})) = -\log C_{EV}(u_1, \ldots, u_d)
\]

\[
\iff \quad \lim_{s \to 0} \frac{1 - C(u_1^s, \ldots, u_d^s)}{s} = -\log C_{EV}(u_1, \ldots, u_d)
\]

\[
\iff \quad \lim_{s \to 0} \frac{1 - C(1 - sw_1, \ldots, 1 - sw_d)}{s} = \frac{1}{s}
\]

\[
\iff \quad w_i = -\log u_i, \quad u_i^s \approx 1 - sw_i
\]

\[
\iff \quad \lim_{s \to 0} \frac{1 - C(1 - sw_1, \ldots, 1 - sw_d)}{s} = \log C_{EV}(e^{-w_1}, \ldots, e^{-w_d})
\]

Introduce the exponent function

\[
a(w_1, \ldots, w_d) := \lim_{s \to 0} \frac{1 - C(1 - sw_1, \ldots, 1 - sw_d)}{s}.
\]
Tail Dependence $\Rightarrow$ Exponent Function

- Let $U_i = F_i(X_{i,n})$, $1 \leq i \leq d$, denote the standard uniform random variables such that $(U_1, \ldots, U_d)$ has the distribution $C$.

- Rewrite the exponent function

$$a(w_1, \ldots, w_d) : = \lim_{s \to 0} \frac{1 - C(1 - sw_1, \ldots, 1 - sw_d)}{s}$$

$$= \lim_{s \to 0} \frac{P(U_i > 1 - sw_i, \text{ for at least one } i)}{s}$$

- The copula domain of attraction

$$\lim_{n \to \infty} C^n(u_{1/n}^1, \ldots, u_{1/n}^d) = C_{EV}(u_1, \ldots, u_d)$$

$$\iff a(w_1, \ldots, w_d) = - \log C_{EV}(e^{-w_1}, \ldots, e^{-w_d})$$

$$\iff C_{EV}(u_1, \ldots, u_d) = \exp \left\{ - a(- \log u_1, \ldots, - \log u_d) \right\}$$

where the exponent function $a(\cdot)$ can be determined by tail dependence of copula $C$. 
Example: Gumbel Copula

Consider the bivariate Gumbel copula:

\[ C(u_1, u_2; \delta) = \exp \left\{ -\left[ (-\log u_1)^\delta + (-\log u_2)^\delta \right]^{1/\delta} \right\}, \ 0 \leq u_1, u_2 \leq 1, \]

where \( \delta \geq 1 \) is a parameter. Then for any \( t > 0 \),

\[ C(u_1^t, u_2^t; \delta) = C^t(u_1, u_2; \delta), \ 0 \leq u_1, u_2 \leq 1. \]

The Gumbel copula is an extreme value copula.

The exponent function of a bivariate Gumbel copula is

\[ a(w_1, w_2) = (w_1^\delta + w_2^\delta)^{1/\delta} \]

The extreme value copula of a Gumbel copula is the same as itself:

\[ C_{EV}(u_1, u_2) = \exp \left\{ - a(-\log u_1, -\log u_2) \right\} = C(u_1, u_2; \delta). \]
Example: Galambos Copula

Consider a bivariate Galambos copula: for \( 0 \leq u_1, u_2 \leq 1 \),

\[
C(u_1, u_2; \theta) = u_1 u_2 \exp \left\{ \left[ ( - \log u_1 )^{-\delta} + ( - \log u_2 )^{-\delta} \right]^{-1/\delta} \right\},
\]

which satisfies the multivariate scaling property.

The exponent function of a bivariate Galambos copula is

\[
a(w_1, w_2) = w_1 + w_2 - ( w_1^{-\delta} + w_2^{-\delta} )^{-1/\delta}.
\]

The extreme value copula of a Galambos copula is the same as itself:

\[
C_{\text{EV}}(u_1, u_2) = \exp \left\{ - a( - \log u_1, - \log u_2 ) \right\} = C(u_1, u_2; \delta).
\]
Consider a bivariate Clayton copula:

\[ C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \quad \forall (u_1, u_2) \in [0, 1]^2, \]

which does not satisfy the multivariate scaling property.

The exponent function of a bivariate Clayton copula is

\[ a(w_1, w_2) = w_1 + w_2. \]

The extreme value copula of a bivariate Clayton copula is the independent copula:

\[ C_{EV}(u_1, u_2) = u_1 u_2. \]
Multivariate Scaling \Rightarrow \text{Pickands Representation}

- Let $S_{d-1}^d = \{ \mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}_+^d, ||\mathbf{a}||_1 = \sum_{i=1}^d a_i = 1 \}$ denote the unit ball in $\mathbb{R}_+^d$.
- We use the $l_1$ norm $|| \cdot ||_1$ here, but any norm would be all right.

\begin{center}
Pickands Representation
\end{center}

For any $(u_1, \ldots, u_d) \in [0, 1]^d$, 

$$C_{EV}(u_1, \ldots, u_d) = \exp \left\{ -c \int_{S_{d-1}^d} \max_{1 \leq i \leq d} \{ -a_i \log u_i \} dQ(\mathbf{a}) \right\},$$

where $c$ is normalizing constant and $Q$ is a probability distribution defined on $S_{d-1}^d$ such that $c \int_{S_{d-1}^d} a_i dQ(\mathbf{a}) = 1, 1 \leq i \leq d$. The distribution $Q(\cdot)$, called the spectral distribution, depends only on copula $C$. 

Different Norms

locus of $\mathbf{x}$ such that $\|\mathbf{x}\|_{\text{sup}} = 1$

locus of $\mathbf{x}$ such that $\|\mathbf{x}\|_{2} = 1$

locus of $\mathbf{x}$ such that $\|\mathbf{x}\|_{1} = 1$
The estimation and asymptotic properties of the probability distribution $Q(\cdot)$ can be found in Resnick (2007).

- Any probability distribution $Q(\cdot)$ on $\mathbb{S}^{d-1}_+$ can be approximated via discrete probability distribution on $\mathbb{S}^{d-1}_+$.

- The discretization of $Q(\cdot)$ leads to a rich parametric family of max-stable multivariate Fréchet distributions (min-stable multivariate exponential distributions, including the well-known Marshall-Olkin distribution).

- The discretization of $Q(\cdot)$ leads to a rich parametric family of EV copulas with singularity components (e.g., Lévy-frailty copulas, Mai and Scherer, 2009).
Pickands Representation ⇒ Positive Association

- \((X_1, \ldots, X_d) \sim H\) is said to positively associated if
  \[
  E\left[f(X_1, \ldots, X_d)g(X_1, \ldots, X_d)\right] \geq Ef(X_1, \ldots, X_d)Eg(X_1, \ldots, X_d),
  \]
  \[\forall f, g : \mathbb{R}^d \rightarrow \mathbb{R}, \text{ non-decreasing}.\]

- The positive association is a strong positive dependence and trivially implies that \((X_1, \ldots, X_d)\) is positively correlated pair-wise.

- The positive association, equivalent to FKG inequality widely used in statistical physics (Fortuin, Kastelyn, and Ginibre, 1971), is one of several basic inequalities in analyzing concentration phenomena (Ledoux and Talagrand, 1991).
Pickands Representation $\Rightarrow$ Positive Association

- $(X_1, \ldots, X_d) \sim H$ is said to positively associated if

$$E[f(X_1, \ldots, X_d)g(X_1, \ldots, X_d)] \geq Ef(X_1, \ldots, X_d)Eg(X_1, \ldots, X_d),$$

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**Theorem (Marshall and Olkin, 1983)**

The MEV distribution $H$ is positively associated.