Outline

1. Brownian Motion
2. Option Pricing
3. Asset Pricing with Binomial Trees
Stochastic Processes

A stochastic process $X := (X_t, t \in T)$ is a collection of random variables defined on some sample space $\Omega$, indexed by $t \in T \subseteq \mathbb{R}$.

If index set $T$ is a finite or countably infinite set, $X$ is said to be a discrete-time process. If $T$ is an interval, then $X$ is a continuous-time process.

A stochastic process $X$ is a (measurable) function of two variables: time $t$ and sample point $\omega$.

Fix time $t$, $X_t = X_t(\omega), \omega \in \Omega$, is a random variable.

Fix sample point $\omega$, $X_t = X_t(\omega), t \in T$, is a sample path.
Brownian Motion

A stochastic process \( B = (B_t, t \geq 0) \) is called a standard Brownian motion or a Wiener process if

- \( B_0 = 0 \),
- for every \( t > 0 \), \( B_t \) has a normal \( N(0, t) \) distribution,
- it has continuous sample paths, and
- it has stationary, independent increments. That is, for any \( 0 \leq s \leq t \) and \( h \geq 0 \),

\[
B_t - B_s \overset{d}{=} B_{t+h} - B_{s+h}, \quad (\overset{d}{=} \text{means “same distribution”})
\]

and for any \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \),

\[
B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_k} - B_{t_{k-1}}, \text{ are independent.}
\]
Various Sample Paths of Brownian Motion

Figure: Left = price, Right = log-returns
3-Dimensional Brownian Motion
Brownian motion is named after the botanist Robert Brown who first observed, in 1827, the irregular motion of pollen grains immersed in water. By the end of the 19th century, the phenomenon was understood as a result of molecular (or so-called Brownian particles) bombardment, but was lacking a satisfactory theoretical explanation.

In 1900, Louis Bachelier had employed it to model the stock market, where the analogue of molecular bombardment is the interplay of the myriad of individual market decisions that determine the market price. The construction in Bachelier’s PhD thesis (Théorie de la Spéculatiion) was far from being mathematically rigorous but captured many properties of the process.
Early History of Continuum Probability

- In 1905, Albert Einstein predicted the existence of Brownian motion by introducing the density of Brownian particles that satisfies a diffusion equation (a parabolic PDE), and provided a way to empirically verify the reality of the atom.

- Jean Perrin verified experimentally Einstein’s explanation of Brownian motion and thereby confirmed the atomic nature of matter. The Nobel Prize for Physics was awarded to Perrin in 1926 for this achievement.

- Norbert Wiener (1923) was the first to put Brownian motion on a firm mathematical basis. For quite some time, Brownian motion was called the Bachelier-Wiener process among some mathematicians.

- The Bachelier heritage includes his work on diffusion processes, which at least partially motivated Kolmogorov to develop the analytic theory of continuous Markov processes.
Elementary Properties of Brownian Motion

- For any $t > s$, increment stationarity implies that

$$B_t - B_s \overset{d}{=} B_{t-s} - B_0 = B_{t-s} \sim N(0, t - s).$$

That is, the larger the interval, the larger the fluctuations of $B$ on this interval.

- Reflection Symmetry:

$$( -B_t, t \geq 0 ) \overset{d}{=} (B_t, t \geq 0)$$

is also a standard Brownian Motion.

- $\mu_B(t) := E(B_t) = 0$, and for any $t > s$, the covariance function

$$\text{cov}(B_t, B_s) = E(B_t B_s) - E(B_t)E(B_s) = E[((B_t - B_s) + B_s)B_s]$$

$$= E[(B_t - B_s)B_s] + E(B_s^2) = E(B_t - B_s)E(B_s) + s = \min(s, t).$$
Self-Similarity ⇒ Nowhere Differentiable

- Brownian motion is 0.5-self-similar. That is,

\[(s^{1/2}B_{t_1}, \ldots, s^{1/2}B_{t_n}) \overset{d}{=} (B_{st_1}, \ldots, B_{st_n})\]

spatial rescaling by \(s^{1/2}\)  

time rescaling by \(s\)

for every \(s > 0\) and any choice of \(t_i \geq 0, \ i = 1, \ldots, n\). In particular,

\[s^{1/2}B_t \overset{d}{=} B_{st}, \ \forall \ s \geq 0, \ t \geq 0.\]

- Self-similarity means that the properly scaled patterns of a sample path in any small or large time interval have a similar shape.

- Brownian sample paths are nowhere differentiable. That is, any sample path changes its shape in the neighborhood of any time epoch in a completely non-predictable fashion (Wiener, Paley and Zygmund, 1930s).
Ethernet traffic burstiness at different time scales

- Time Unit = 100 Seconds
- Time Unit = 10 Seconds
- Time Unit = 1 Second
- Time Unit = 0.1 Second
Unbounded Variation

Brownian motion has unbounded variation; that is,

$$\sup_{\tau} \sum_{i=1}^{n} |B_{t_i}(\omega) - B_{t_{i-1}}(\omega)| = \infty, \text{ for almost all } \omega,$$

where the supremum is taken over all possible partitions \( \tau : 0 = t_0 < \cdots < t_n = T \) of any finite interval \([0, T]\).

The Rise of Stochastic Calculus (Since 1950s ...)

The unbounded variation and non-differentiability of Brownian sample paths are major reasons for the failure of classical integration methods, when applied to these paths, and for the introduction of stochastic calculus (Kiyoshi Itô, Ruslan Stratonovich, Paul Malliavin, ...).
Reflection Principle of Brownian Motion \((B_t, t \geq 0)\)

- Define the first passage time
  \[ T_b = \inf\{ t \geq 0 : B_t = b \}, \quad b > 0. \]
- \(T_b\) is a random time with
  \[ P(T_b \leq t) = P(B_s = b \text{ for some } 0 < s \leq t) = P(\max_{0 \leq s \leq t} B_s \geq b). \]
- \[ P(B_t \leq b | T_b \leq t) = P(B_t \geq b | T_b \leq t) = \frac{1}{2}. \]

**Figure**: A sample path for event \(\{T_b \leq t\}\) with \(b = 0.5, t = 4.0\)
Extremes of Brownian Motion \((B_t, t \geq 0)\)

- Let \(M_t = \max_{0 \leq s \leq t} B_s\) be the running maximum of Brownian motion.
- Since \(\{B_t \geq b\} \subseteq \{T_b \leq t\}\), we have

\[
\frac{1}{2} = P(B_t \geq b | T_b \leq t) = \frac{P(B_t \geq b)}{P(T_b \leq t)}
\]

Thus the tail probability

\[
P(M_t \geq b) = P(T_b \leq t) = 2P(B_t \geq b) = 2\Phi\left(-\frac{b}{\sqrt{t}}\right)
\]

which decays exponentially fast to zero as the threshold \(b \to +\infty\).

- Brownian motion “starts afresh” at any fixed time epoch (Markov Property).
- Brownian motion also “starts afresh” at certain random time epoch, such as \(T_b\) (Strong Markov Property).
Brownian Motion with Drift

Let $B = (B_t, t \geq 0)$ denote Brownian Motion.

- The process
  \[ X := (X_t, t \geq 0) = (\mu t + \sigma B_t, t \geq 0), \]
  for constants $\sigma > 0$ and $\mu \in \mathbb{R}$, is called a Brownian motion with (linear) drift.

- The mean and covariance function:
  \[ \mu_X(t) = \mu t, \quad \text{cov}(X_t, X_s) = \sigma^2 \min(t, s), \quad s, t \geq 0. \]
Brownian Motion with Drift
Let $S_t$ denote the price of a **risky asset** (let’s call it a stock) at time $t$.

Assume that the **relative return** from the asset in the period of time $[t, t + dt]$ has a linear drift trend $c \, dt$ which is disturbed by a stochastic noise term $\sigma dB_t$.

$$\frac{S_{t+dt} - S_t}{S_t} = \text{BM with drift } c \, dt + \sigma dB_t,$$

or $dS_t = cS_t \, dt + \sigma S_t \, dB_t$.

The constant $c > 0$ is the so-called **mean rate of return**, and $\sigma > 0$ is the **volatility**.

Observe that this is a **crude, first order approximation** to a real price process. But people in economics believe in exponential growth and they are often happy with this model.
Geometric Brownian Motion

- Solve the equation using Itô Lemma, we obtain that

\[ S_t = S_0 e^{(c-\sigma^2/2)t + \sigma B_t}, \quad t \geq 0, \]

for constants \( c > 0 \) and \( \sigma > 0 \). This process called a geometric Brownian motion with drift \( \mu = c - \sigma^2/2 \).

- For fixed \( t \), \( S_t \) has a log-normal distribution, and

\[ S_t \sim \text{LogN}(\log S_0 + (c - \sigma^2/2)t, \sigma^2 t). \]

- Geometric Brownian Motion is used to model stock prices in the Black-Scholes model (Black and Scholes, 1973) and is the most widely used model for stock price behavior.
Two Sample Paths of Geometric Brownian Motion

**Figure**: Blue: $c = 1.02, \sigma^2 = 0.04$; Green: $c = 0.625, \sigma^2 = 0.25$
A call option at time $t = 0$ is a “ticket” which entitles you to buy one share of stock until or at time $T$, the time of maturity or time of expiration of the option.

If you can exercise this option (or exercise the call) at a fixed price $K$, called the exercise price or strike price of the option, only at time of maturity $T$, this is called a **European call option**. If you can exercise it until or at time $T$, it is called an **American call option**. There are many other kinds ....

The purchaser of a European call option is entitled to a payment of

$$X = (S_T - K)^+ = \max(0, S_T - K).$$

A put is an option to sell stock at a strike price $K$ on or until a particular date of maturity $T$. A **European put option** is exercised only at time of maturity with profit $X = (K - S_T)^+$, and an **American put** can be exercised until or at time $T$. 
Arbitrage Free Pricing

How much would you be willing to pay for such a ticket, i.e. what is a rational price $v$ for this option at time $t = 0$?

Black, Scholes and Merton responded as follows:

1. After investing this rational value $v$ of money in bond at time $t = 0$ at risk-free interest rate $r$, you can manage your portfolio so as to yield the same payoff $X = (S_T - K)^+$ as if the option had been purchased. That is,

$$v = E^*[e^{-rT}X] = E^*[e^{-rT}(S_T - K)^+]$$

where $S_t^* \sim LogN(\log S_0 + (r - \sigma^2/2)t, \sigma^2 t)$.

2. If the option were offered at any price other than this rational value, there would be an opportunity of arbitrage, i.e. for unbounded profits without an accompanying risk of loss.
Let $d$ and $u$ be the positive constants such that $d < 1 < u$.

- Divide the time interval $[0, T]$ into $n$ pieces of equal length $\Delta t = T/n$.
- $S_k =$ price of a certain stock at the end of the $k$-th time period.
- At the next time step, the price moves up to $S_k u$ with probability $p$.
- At the next time step, the price moves down to $S_k d$ with probability $1 - p$.
- The initial stock price $S_0 = S$.
- Find the rational price $\nu$ for one share of an option under the arbitrage free principle.
Special Case: $T = 1$

The payoff of the option

$$X = \begin{cases} 
C_u & \text{if } S_1 = S_0u \\
C_d & \text{if } S_1 = S_0d.
\end{cases}$$

One dollar at $t = 0$ is worth $R$ dollars at $t = 1$ (due to interest from risk-free assets). Because of the arbitrage free principle,

$$d \leq R \leq u.$$
Special Case: $T = 1$

- Our portfolio consists of one share of the option and $x$ shares of the stock. At $t = 0$, the value of the portfolio is $v + xS_0$.
- At maturity $t = 1$, we must have

$$x(S_0u) + C_u = x(S_0d) + C_d,$$

that is, $x = \frac{C_d - C_u}{S_0u - S_0d}$,

no matter how the stock moves (risk-neutral). On the other hand, we must also have

$$(v + xS_0)R = x(S_0u) + C_u$$

due to the arbitrage free principle.
- These equations yield the rational price

$$v = \frac{1}{R} \left( \frac{R - d}{u - d} C_u + \frac{u - R}{u - d} C_d \right).$$
Risk-Neutral Probability

Let

\[ p^* = \frac{R - d}{u - d} \geq 0 \text{ and } q^* = \frac{u - R}{u - d} \geq 0. \]

Observe that \( p^* + q^* = 1 \). The probability vector \((p^*, q^*)\) is called the risk-neutral probabilities.

The option price (or its rational value) is rewritten as

\[ v = \frac{1}{R} (p^* C_u + q^* C_d) = E^*[R^{-1} X], \]

where \( E^* (\cdot) \) is the expected value with respect to the risk-neutral probabilities.

The Key Idea

The option value at \( t = 0 \) is the expectation of discounted option value at \( t = 1 \) with respect to the risk-neutral probability measure \((p^*, q^*)\).
Under the risk-neutral probability measure \((p^*, q^*)\), the average growth rate of the stock price equals the risk-free interest rate \(R\):

\[
E^*(S_1) = p^* S_0 u + q^* S_0 d = RS_0.
\]

The physical probabilities \((p, 1-p)\) of the stock price movement are not directly related to the option pricing; they are related to the option value only through the interest rate \(R\).
Multiperiod Model (Cox, Ross and Rubinstein, 1979)

Rational Value at $t = 0$ with Payoff $X$ at Maturity $T = n \Delta t$

$$v = \mathbb{E}_* [R^{-n} X]$$

Figure: Multi-Period Binomial Tree $n = 3$
Black-Scholes Model

Assume that the stock price follows a geometric Brownian motion:

\[ S_t = S_0 e^{(c - \sigma^2/2)t + \sigma B_t}, \quad t \geq 0, \]

that is,

\[ S_t \sim \text{LogN}\left(\log S_0 + (c - \sigma^2/2)t, \sigma^2 t\right), \]

where the drift \( c \) and volatility \( \sigma \) are the (physical) parameters of the stock price process.

The risk-free interest rate is given by \( r \).

Using the binomial approximation or martingale method, it can be shown that the risk-neutral probability measure is given by the log-normal distribution

\[ S^*_t \sim \text{LogN}\left(\log S_0 + (r - \sigma^2/2)t, \sigma^2 t\right), \]

Rational Value at \( t = 0 \) with Payoff \( X \) at Maturity \( T \)

\[ v = E^*_\left[e^{-rT}X\right] \]
Example: European Call Option (BLS_Call)

You can buy one share of stock at a fixed price $K$ only at time of maturity $T$.

- $X = (S_T - K)^+ = \max\{0, S_T - K\}$.
- Calculate $\nu = E[e^{-rT}(S_T - K)^+]$, where

$$S_T \sim \text{LogN}(\log S_0 + (r - \sigma^2/2)T, \sigma^2T).$$

Black-Scholes Formula (Fisher Black and Myron Scholes, 1973)

$$\nu = S_0 \Phi(\sigma\sqrt{T} - \theta) - Ke^{-rT}\Phi(-\theta)$$

where $\Phi(x)$ is the standard normal CDF and

$$\theta = \frac{1}{\sigma\sqrt{T}} \log \frac{K}{S_0} + \left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)\sqrt{T}.$$
Example: European Put Option (BLS_Put)

You can sell one share of stock at a fixed price $K$ only at time of maturity $T$.

- $X = (K - S_T)^+ = \max\{0, K - S_T\}$.
- Calculate $v = E[e^{-rT}(K - S_T)^+]$, where

$$S_T \sim \text{LogN}(\log S_0 + (r - \sigma^2/2)T, \sigma^2 T).$$

Put-Call Parity

- BLS_Call - BLS_Put = $E[e^{-rT}(S_T - K)] = S_0 - e^{-rT}K$.
- $v = Ke^{-rT}\Phi(\theta) - S_0\Phi(\theta - \sigma\sqrt{T})$, where $\Phi(x)$ is the standard normal CDF and

$$\theta = \frac{1}{\sigma\sqrt{T}} \log \frac{K}{S_0} + \left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)\sqrt{T}.$$
Example: Lookback Call Option

- \( X = \left( \max_{0 \leq t \leq T} S_t - K \right)^+ = \max \{ 0, \max_{0 \leq t \leq T} S_t - K \} \).
- Calculate \( v = E \left[ e^{-rT} \left( \max_{0 \leq t \leq T} S_t - K \right)^+ \right] \), where

\[ S_T \sim \text{LogN}(\log S_0 + (r - \sigma^2/2)T, \sigma^2 T). \]

- After some tedious calculation, we have

\[ v = \text{BLS}_\text{Call} + \frac{\sigma^2}{2r} S_0 \left[ \Phi(\theta_+) - e^{-rT} \left( \frac{K}{S_0} \right)^{2r/\sigma^2} \Phi(\theta_-) \right] \]

extra premium

where \( \Phi(x) \) is the standard normal CDF and

\[ \theta_\pm = \frac{1}{\sigma \sqrt{T}} \log \frac{S_0}{K} + \left( \frac{\sigma}{2} \pm \frac{r}{\sigma} \right) \sqrt{T}. \]