Math 416/516: Stochastic Simulation

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Week 14
1 Sensitivity Analysis
Hedging Against Risk

- Let $S_t$ denote the price of the underlying risky asset (e.g., stock) at $t$. For example, $S_t$ follows a geometric Brownian motion:

$$S_t = S_0 e^{(r - \sigma^2/2)t + \sigma B_t}, \quad 0 \leq t \leq T,$$

where $r =$ interest rate, $\sigma =$ volatility, $S_0 =$ initial value.

- Let $V(S, r, \sigma, t)$ denote the value of the option at time $t \in [0, T]$.

The Feynman-Kac Formula (also see Harrison and Pliska, 1981)

$$V(S, r, \sigma, t) = \mathbb{E}(e^{-r(T-t)}h(S_T) \mid S_t = S_0), \quad 0 \leq t \leq T.$$

- Hedge one share of the option with $x$ shares of the risky asset:

$$\text{Value of your portfolio at } t = V(S, r, \sigma, t) + xS.$$

- Set the derivative (with respect to $S$) to zero, and we have

$$x = -\frac{\partial}{\partial S} V(S, r, \sigma, t).$$
Greeks

- \( \Delta := \frac{\partial V}{\partial S}; \)
- \( \Gamma := \frac{\partial^2 V}{\partial S^2}; \)
- \( \rho := \frac{\partial V}{\partial r}; \)
- \( \Theta := \frac{\partial V}{\partial t}; \)
- \( \zeta := \frac{\partial V}{\partial \sigma} \) (also known as Vega or kappa).

As with Gamma hedging, one can zeta hedge to reduce sensitivity to the volatility. Zeta is a somewhat different from the other Greeks since it is a derivative with respect to a parameter \( \sigma \), which may have been assumed to be constant (not a variable).

The volatility \( \sigma \) is a model quantity that is not known very accurately, and so using zeta is a major step towards eliminating some model risk.
Example: European Call Option (BLS_Call)

- Assume that the stock price follows a geometric Brownian motion:
  \[ S_t = S_0 e^{(r - \sigma^2/2)t + \sigma B_t}, \quad 0 \leq t \leq T, \]
  where \( r \) = interest rate, and \( \sigma \) = volatility of the stock price process.

- Calculate \( v = E[e^{-rT}(S_T - K)^+] \), via Black-Scholes Formula:
  \[ v = S_0 \Phi(\sigma \sqrt{T} - b) - Ke^{-rT} \Phi(-b) \]
  where \( \Phi(x) \) is the standard normal CDF and
  \[ b = \frac{1}{\sigma \sqrt{T}} \log \frac{K}{S_0} + \left( \frac{\sigma}{2} - \frac{r}{\sigma} \right) \sqrt{T}. \]

- \( \Delta = \Phi(\sigma \sqrt{T} - b) \).
BLS_Call: $K = 60, \sigma = 0.29, r = 0.04, T = 0.3 \text{ (year)}$
Monte Carlo Simulation of Greeks

- $X(\theta) =$ discounted payoff, where $\theta$ is the variable or parameter of the interest.

- The performance measure of the interest (e.g., the value of the option):
  \[ V(\theta) = E[X(\theta)], \]
  where the expected value is taken under some probability distribution that does not depend on $\theta$ (e.g., the standard Brownian motion).

- Under some regularity conditions, we can exchange the order of expectation and differentiation:
  \[ V'(\theta) = \frac{\partial}{\partial\theta} E[X(\theta)] \]
  Method of Finite Difference
  \[ = E \left[ \frac{\partial}{\partial\theta} X(\theta) \right] \]
  Pathwise Differentiation

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Method of Finite Difference

- Write

\[ V'(\theta) \approx \frac{V(\theta + h) - V(\theta - h)}{2h}, \quad \text{as } h \to 0. \]

- Estimate \( V(\theta + h) \) and \( V(\theta - h) \) using unbiased MC estimators \( \hat{V}(\theta + h) \) and \( \hat{V}(\theta - h) \).

- Estimate \( V'(\theta) \) via the estimator

\[ \hat{V}'(\theta, h) = \frac{\hat{V}(\theta + h) - \hat{V}(\theta - h)}{2h}. \]

- Since

\[
\text{var}(\hat{V}'(\theta, h)) = \frac{1}{4h^2}[\text{var}(\hat{V}(\theta + h)) + \text{var}(\hat{V}(\theta - h))]
\]

\[
-2\text{cov}(\hat{V}(\theta + h), \hat{V}(\theta - h))
\]

- To reduce the variance, common random numbers should be used in \( \hat{V}(\theta + h) \) and \( \hat{V}(\theta - h) \) to maximize \( \text{cov}(\hat{V}(\theta + h), \hat{V}(\theta - h)) \).
Remarks

Two Problems for the finite difference method:

- If $h$ is small, $\text{var}(\hat{V}'(\theta, h))$ may be unbounded; that is, reducing $h$ will not make the estimate more accurate, unless the simulation simple size is increased significantly.

- Even if both $\hat{V}(\theta + h)$ and $\hat{V}(\theta - h)$ are unbiased, the estimator $\text{var}(\hat{V}'(\theta, h))$ may still be biased because $V(\theta)$ could be a non-linear function of $\theta$. Using Taylor’s expansion,

$$\text{bias} = \frac{V(\theta + h) - V(\theta - h)}{2h} - V'(\theta) \approx \frac{1}{6} V'''(\theta) h^2.$$  

The bias can be decreased for non-convex (non-concave) function $V(\theta)$ when $h$ becomes smaller. But this would increase the variance.

- The issues become more significant for higher-order derivatives.
Example: European Call Option (BLS_Call)

- Assume that the stock price follows a geometric Brownian motion:
  \[ S_t = S_0 e^{(r - \sigma^2/2)t + \sigma B_t}, \quad t \geq 0, \]

  where \( r \) = interest rate, and \( \sigma \) = volatility of the stock price process.

- The Delta of \( \nu = E[e^{-rT}(S_T - K)^+] \) is
  \[ \Delta = \Phi(\sigma \sqrt{T} - b), \]

  where \( \Phi(x) \) is the standard normal CDF and

  \[ b = \frac{1}{\sigma \sqrt{T}} \log \frac{K}{S_0} + \left( \frac{\sigma}{2} - \frac{r}{\sigma} \right) \sqrt{T}. \]

- We can also estimate \( \Delta \) via MC simulation. Let \( \theta = S_0 \), the initial value of the stock price.
Example: Finite-Difference Algorithm (BLS_Call)

Consider \( \Delta = \frac{\partial}{\partial \theta} E(X(\theta)) \), where

\[
X(\theta) = V(S_0) = e^{-rT} \left[ S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T} Z} - K \right]^+, \quad Z \sim N(0, 1).
\]

1. Generate \( Z \sim N(0, 1) \), and for a small \( h \), set

\[
X^h = (S_0 + h) e^{(r-\sigma^2/2)T + \sigma \sqrt{T} Z} \quad \text{and} \quad X^{-h} = (S_0 - h) e^{(r-\sigma^2/2)T + \sigma \sqrt{T} Z}
\]

where the same \( Z \) is used to minimize the variance of the difference.

2. Set

\[
H = \frac{1}{2h} \left[ e^{-rT} (X^h - K)^+ - e^{-rT} (X^{-h} - K)^+ \right].
\]

3. Repeat (1) and (2) \( n \) times to generate \( n \) values \( H_1, \ldots, H_n \). The estimate of \( \Delta \) is given by \( \hat{\Delta} = \frac{1}{n} \sum_{i=1}^{n} H_i \).
\[ S_0 = 50, \ r = 0.05, \ \sigma = 0.2, \ T = 1, \ n = 10,000,000, \]
and estimate the delta for \( K = 50 \) and \( 55 \), respectively.

Table 10.2: Finite difference method: delta for call options

<table>
<thead>
<tr>
<th></th>
<th>( K = 50 ) (( \Delta = 0.6368 ))</th>
<th></th>
<th>( K = 55 ) (( \Delta = 0.4496 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( h = 1 )</td>
<td>( h = 0.1 )</td>
<td>( h = 0.01 )</td>
</tr>
<tr>
<td>Estimate</td>
<td>0.6364</td>
<td>0.6368</td>
<td>0.6366</td>
</tr>
<tr>
<td>S.E.</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0002</td>
</tr>
</tbody>
</table>
Pathwise Differentiation

Under some regularity conditions,

\[ V'(\theta) = E \left[ \frac{\partial}{\partial \theta} X(\theta) \right]. \]

- \( X(\theta) \) is a random variable, and so \( X'(\theta) := \frac{\partial}{\partial \theta} X(\theta) \) is also a random variable.
- Let \( X'_1, \ldots, X'_n \) be i.i.d. samples drawn from the distribution of \( X'(\theta) \), then the estimator of \( X'(\theta) \) is

\[ \hat{X}'(\theta) = \frac{1}{n} \sum_{i=1}^{n} X'_i. \]

- Advantage: When \( X(\theta) \) has an explicit stochastic representation, \( \frac{\partial}{\partial \theta} X(\theta) \) also has its explicit stochastic representation so that MC simulation is straightforward. The estimates of Greeks do not involve finite differencing.
Example: European Call Option (BLS_Call)

- Assume that the stock price follows a geometric Brownian motion:
  \[ S_t = S_0 e^{(r - \sigma^2/2)t + \sigma B_t}, \quad 0 \leq t \leq T, \]
  where \( r \) = interest rate, and \( \sigma \) = volatility of the stock price process.
- Let \( \theta = S_0 \). The Delta of \( \nu = E[e^{-rT}(S_T - K)^+] \) is
  \[ \Delta = E \left[ \frac{\partial}{\partial S_0} e^{-rT}(S_T - K)^+] \right] = E \left[ \frac{\partial}{\partial S_0} e^{-rT}(S_T - K) I\{S_T \geq K\} \right] \]
  Since
  \[ \frac{\partial}{\partial S_0} e^{-rT}(S_T - K) I\{S_T \geq K\} = I\{S_T \geq K\} \frac{e^{-rT} S_T}{S_0}, \]
  \[ \Delta = E \left[ I\{S_T \geq K\} \frac{e^{-rT} S_T}{S_0} \right]. \]
Example: Pathwise Differentiation Algorithm

1. Generate $Z \sim N(0, 1)$, and set

$$X = e^{(r - \sigma^2/2)T + \sigma \sqrt{T} Z}$$

2. Set

$$H = I\{S_0 X \geq K\} e^{-rT} X.$$ 

3. Repeat (1) and (2) $n$ times to generate $n$ values $H_1, \ldots, H_n$. The estimate of $\Delta$ is given by

$$\hat{\Delta} = \frac{1}{n} \sum_{i=1}^{n} H_i.$$
Example: BLS_Call via Pathwise Differentiation

Use

\[ S_0 = \$50, \ r = 0.05, \ \sigma = 0.2, \ T = 1, \ n = 10,000. \]

<table>
<thead>
<tr>
<th>Strike price K</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical value</td>
<td>0.9286</td>
<td>0.8097</td>
<td>0.6368</td>
<td>0.4496</td>
<td>0.2872</td>
</tr>
<tr>
<td>Estimate</td>
<td>0.9244</td>
<td>0.8064</td>
<td>0.6373</td>
<td>0.4469</td>
<td>0.2867</td>
</tr>
<tr>
<td>S.E.</td>
<td>0.0036</td>
<td>0.0049</td>
<td>0.0058</td>
<td>0.0059</td>
<td>0.0054</td>
</tr>
</tbody>
</table>

**Figure**: Theoretical value is obtained via Black-Scholes Formula, and estimates are done via Pathwise Differentiation.
Cautionary Remark

- Due to its strong regularity conditions, path differentiation method has to be used with care. For example, consider the Delta at $t = 0$ for a binary option with maturity $T$ and discounted payoff

$$X(S_0) = e^{-rT} I\{S_T \geq K\}.$$  

It is easy to see that $\frac{\partial X(S_0)}{\partial S_0} = 0$, but the Delta of this option is clearly not zero.

- Paul Malliavin introduced in 1978 the stochastic derivative of a random variable, commonly known as the Malliavin derivative. The Malliavin derivative has been widely used in control theory, portfolio management with partial information, and sensitivity analysis and efficient computation of the “Greeks” in finance.