Math 416/516: Stochastic Simulation

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Week 12
Outline

1. Stochastic Integral

2. Itô Lemmas = Chain Rules in Stochastic Calculus
Integrating With Respect To Brownian Motion

Let $B = (B_t, t \geq 0)$ be Brownian motion.

- **Goal:** Define an integral of type $\int_0^1 f(t)dB_t(\omega)$, where $f(t)$ is a function or a stochastic process on $[0, 1]$ and $B_t(\omega)$ is a Brownian sample path.

- **For example,** how to define and calculate

\[
\int_0^1 B_t dB_t? \quad \text{(Is } \int_0^1 B_t dB_t = \frac{1}{2} B_1^2?)
\]

- **Difficulty:** The path $B_t(\omega)$ does not have a derivative.
Integrating With Respect To a Function

- Consider a partition of the interval \([0, 1]\):
  \[
  \tau_n : 0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = 1, \quad n \geq 1.
  \]

- Let \(f\) and \(g\) be two real-valued functions on \([0, 1]\) and define
  \[\Delta_i g := g(t_i) - g(t_{i-1}), \quad 1 \leq i \leq n,\]
  and the Riemann-Stieltjes sum:
  \[
  S_n = \sum_{i=1}^{n} f(y_i) \Delta_i g = \sum_{i=1}^{n} f(y_i)(g(t_i) - g(t_{i-1})),
  \]
  for \(t_{i-1} \leq y_i \leq t_i, \quad i = 1, \ldots, n.\)

**Definition**

If the limit \(S = \lim_{n \to \infty} S_n\) exists as \(\text{mesh}(\tau_n) \to 0\) and \(S\) is independent of the choice of the partitions \(\tau_n\), and their intermediate values \(y_i\)'s, then \(S\), denoted by \(\int_{0}^{1} f(t) \, dg(t)\), is called the Riemann-Stieltjes integral of \(f\) with respect to \(g\) on \([0, 1]\).
Riemann-Stieltjes Integrals

- When does the Riemann-Stieltjes integral \( \int_0^1 f(t)dg(t) \) exist, and is it possible to take \( g = B \) for Brownian motion \( B \) on \([0, 1]\)?
- One usual assumption is that \( f \) is continuous and \( g \) has bounded variation:
  \[
  \sup_{\tau_n} \sum_{i=1}^{n} \left| g(t_i) - g(t_{i-1}) \right| < \infty.
  \]
- But Brownian sample paths \( B_t(\omega) \) do not have bounded variation.
- We will try to define the integral as a probabilistic average, leading to the Itô Integrals.
A Motivating Example: $\int_0^t B_s dB_s = ?$

Let $B = (B_t, t \geq 0)$ be Brownian motion. Consider a partition of $[0, t]$: 

$\tau_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$, with $\Delta_i = t_i - t_{i-1}$, $n \geq 1$

and the Riemann-Stieltjes sums, for $n \geq 1$,

$$S_n = \sum_{i=1}^{n} B_{t_{i-1}} \Delta_i B,$$

with $\Delta_i B := B_{t_i} - B_{t_{i-1}}$, $1 \leq i \leq n$.

where $t_{i-1}$ is the left end point of $[t_{i-1}, t_i)$.

- After some algebra: $S_n = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_{i=1}^{n} (\Delta_i B)^2 =: \frac{1}{2} B_t^2 - \frac{1}{2} Q_n(t)$,

$$Q_n(t) = \sum_{i=1}^{n} (\Delta_i B)^2.$$

- The limit of $S_n$ boils down to the limit of $Q_n(t)$, as $n \to \infty$. 
Quadratic Variation of Brownian Motion

Consider

\[ Q_n(t) = \sum_{i=1}^{n} (\Delta_i B)^2. \]

- Since Brownian motion has independent and stationary increments, \( E(\Delta_i B) = 0 \) for \( 1 \leq i \leq n \) and
  \[ E(\Delta_i B)^2 = \text{Var}(\Delta_i B) = t_i - t_{i-1} = \Delta_i. \]

- Thus \( E(Q_n(t)) = \sum_{i=1}^{n} E(\Delta_i B)^2 = \sum_{i=1}^{n} \Delta_i = t. \) That is, \( Q_n(t) \) does not converge to 0!

- It can be shown that as \( n \to \infty \),
  \[ \text{var}(Q_n(t)) = E(Q_n(t) - t)^2 \to 0. \]

- From the Chebyshev inequality, we have
  \[ P(|Q_n(t) - t| > \epsilon) \leq \frac{E(Q_n(t) - t)^2}{\epsilon^2} \to 0, \text{ as } n \to \infty. \]
The extra function \( f(t) = t \) is called the quadratic variation of Brownian motion.

Due to self-similarity, Brownian motion exhibits persistent volatility at every scale. This is best illustrated by the fact that

\[
Q_n(t) \to t \neq 0, \quad \text{as } n \to \infty, \text{ in mean square.}
\]

Since \( S_n = \frac{1}{2} B_t^2 - \frac{1}{2} Q_n(t) \) converges in mean square to \( \frac{1}{2} B_t^2 - \frac{1}{2} t \), we define the Itô Integral in the mean square or \( L^2 \) sense:

\[
\int_0^t B_s dB_s := \frac{1}{2} (B_t^2 - t), \quad t \geq 0.
\]

The extra term \( \frac{1}{2} t \) reflects persistent volatility of Brownian motion at micro scales.
Itô Integral \( \int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t) \), \( 0 \leq t \leq 3 \)
General Itô Integral \( \int_0^t C_s dB_s \)

**Basic Assumptions on the Integrand Process \( C \)**

1. \( C = (C_t, 0 \leq t \leq T) \) is adapted to Brownian motion on \([0, T] \), i.e. \( C_t \) is a function of \( B_s, s \leq t \).
2. The integral \( \int_0^T E(C_s^2) ds < \infty \).

For fixed \( t \leq T \) and a given partition \( \tau_n = (t_i) \) of \([0, t] \), we define

\[
\int_0^t C_s dB_s := \lim_{n \to \infty} \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}), \text{ as the limit in mean square.}
\]

That is,

\[
E \left[ \int_0^t C_s dB_s - \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) \right]^2 \to 0, \text{ as } n \to \infty.
\]
Properties

- The Itô stochastic integral has expectation zero and continuous sample paths.
- The Itô stochastic integral satisfies the isometry property:
  \[ E \left( \int_0^t C_s dB_s \right)^2 = \int_0^t E(C_s^2) ds, \quad t \in [0, T]. \]
- For any constants \( c_1 \) and \( c_2 \), and simple processes \( C^{(1)} \) and \( C^{(2)} \) on \([0, T]\),
  \[ \int_0^t (c_1 C_s^{(1)} + c_2 C_s^{(2)}) dB_s = c_1 \int_0^t C_s^{(1)} dB_s + c_2 \int_0^t C_s^{(2)} dB_s. \]
- For any \( t \in [0, T] \),
  \[ \int_0^t C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s. \]
Heuristic Rules

The increment $\Delta_i B = B_{t_i} - B_{t_{i-1}}$ on the interval $[t_{i-1}, t_i]$ satisfies

$$E(\Delta_i B) = 0, \quad \text{Var}(\Delta_i B) = E(\Delta_i B)^2 = \Delta_i = t_i - t_{i-1}.$$ 

- These properties suggest that $(\Delta_i B)^2$ is of order $\Delta_i$.
- In terms of differentials, we can write
  $$(dB_t)^2 = (B_{t+dt} - B_t)^2 \approx dt.$$ 
- In terms of integrals, we write
  $$\int_0^t (dB_s)^2 \approx \int_0^t ds = t.$$ 

First Order Approximation Involving Brownian Motion

$$(dt)^2 \approx 0, \quad (dt)(dB_t) \approx 0, \quad (dB_t)^2 \approx dt.$$ 

These rules can be made mathematically precise in the mean square sense.
Let $B = (B_t, t \geq 0)$ denote Brownian motion. Assume that $f$ is a twice differentiable function, it follows from the Taylor expansion

$$f(B_t + dB_t) - f(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \cdots.$$

- In contrast to the deterministic case, the contribution of the second order term $(dB_t)^2$ in the Taylor expansion is not negligible.
- The squared differential $(dB_t)^2$ can be interpreted as $dt$. 
Let $B = (B_t, t \geq 0)$ be a standard Brownian motion.

- If $f(x)$ is twice continuously differentiable, then, for any $t \geq 0$,

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \frac{1}{2} \int_0^t f''(B_s)ds,$$

or in differential form,

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2} f''(B_t)dt.$$

- If $f(t, x)$ is continuously differentiable with respect to $t$ and twice continuously differentiable with respect to $x$, then

$$df(t, B_t) = \frac{\partial}{\partial t}f(t, B_t)dt + \frac{\partial}{\partial x}f(t, B_t)dB_t + \frac{1}{2} \frac{\partial^2}{\partial x^2}f(t, B_t)dt.$$

extra term
Examples

1. Let $f(t) = t^2$, then

$$B^2_t = 2 \int_0^t B_x dB_x + t$$

resulting in $\int_0^t B_x dB_x = \frac{1}{2}(B^2_t - t)$.

2. Let $f(t) = t^3$, then

$$B^3_t = 3 \int_0^t B^2_s dB_s + 3 \int_0^t B_s ds.$$ 

We cannot express $\int_0^t B_s ds$ in simpler terms of Brownian motion, simulations have to be used.

3. Let $f(t) = e^t$, we have

$$e^{B_t} - e^{B_s} = \int_s^t e^{B_x} dB_x + \frac{1}{2} \int_s^t e^{B_x} dx > \int_s^t e^{B_x} dB_x.$$
Let \((X_t, t \geq 0)\) be a stochastic process, \(\mu(t, X)\) and \(\sigma(t, X)\) two functions.

### A General Modeling Theme

Consider an interval \((t, t + dt)\) at micro scale, \(X_t\) changes at average rate \(\mu(t, X_t) dt\) with volatility \(\sigma(t, X_t)\):

\[
\frac{dX_t - \mu(t, X_t)dt}{\sigma(t, X_t)} \approx \frac{B_{t+dt} - B_t}{\sqrt{dt}} = dB_t
\]

"white noise"

That is, \(X_t\), called Itô’s diffusion, satisfies the SDE:

\[
dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t,
\]

or in integral form:

\[
X_t - X_0 = \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.
\]
Remark

- The random noise $B_t$ is the driving process, and $X_t$ is the solution. Under some regularity conditions on functions $\mu(t, x)$ and $\sigma(t, x)$, the (strong) solution $X_t$ exists. For example, any linear Itô stochastic differential equation given by

$$X_t = X_0 + \int_0^t (c_1 X_s + c_2) \, ds + \int_0^t (\sigma_1 X_s + \sigma_2) \, dB_s,$$

always has the unique strong solution.

- It is possible to replace the driving process $B_t$ by semimartingales, which contains Brownian motion, Poisson process, and a large variety of jump processes.

- They are useful tools when one is interested in modeling the jump character of real-life processes, e.g., crack growth, the strong oscillations of foreign exchange rates or crashes of the stock market.
Consider the linear Itô stochastic differential equation

\[ X_t = X_0 + c \int_0^t X_s \, ds + \sigma \int_0^t X_s \, dB_s. \]

Let \( X_t = f(t, B_t) \), for some smooth function \( f \). From Itô Lemma, we have

\[ cf(t, x) = \frac{\partial f(t, x)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, x)}{\partial x^2}, \quad \sigma f(t, x) = \frac{\partial f(t, x)}{\partial x}. \]

If \( f(t, x) = g(t)h(x) \) is separable, then we have

\[ f(t, x) = g(0)h(0)e^{(c-0.5\sigma^2)t+\sigma x}. \]

The unique strong solution is given by the geometric Brownian motion:

\[ X_t = f(t, B_t) = X_0 e^{(c-0.5\sigma^2)t+\sigma B_t}. \]
Example

Let $S_t$ denote the price of a stock at time $t$.

Assume that the relative return from the stock in the period of time $[t, t + dt]$ has a linear drift trend $c\, dt$ which is disturbed by a stochastic noise term $\sigma dB_t$.

$$\frac{S_{t+dt} - S_t}{S_t} = \text{BM with drift} \left( c\, dt + \sigma dB_t \right), \text{ or } dS_t = cS_t\, dt + \sigma S_t\, dB_t.$$ 

$c = \text{mean rate of return, and } \sigma = \text{volatility}$.

Solve the equation using Itô Lemma, we obtain that

$$S_t = S_0 e^{(c-\sigma^2/2)t + \sigma B_t}, \quad t \geq 0.$$ 

For fixed $t$, $S_t$ has a log-normal distribution, and

$$S_t \sim \text{LogN}(\log S_0 + (c - \sigma^2/2)t, \sigma^2 t).$$
Langevin Equation (used in statistical physics)

### Linear SDE with Additive Noise

\[
X_t = X_0 + c \int_0^t X_s ds + \sigma \int_0^t dB_s, \quad t \in [0, T],
\]

where \( c \) is a constant.

- In the differential form \( dX_t = cX_t dt + \sigma dB_t \).
- This resembles a time series (autoregressive process of order 1, AR(1)),

\[
X_{t+1} - X_t = cX_t + \sigma (B_{t+1} - B_t), \quad \text{or} \quad X_{t+1} = \phi X_t + Z_t,
\]

where \( \phi = c + 1 \) and \( Z_t = \sigma (B_{t+1} - B_t) \sim N(0, \sigma^2) \). This time series model can be considered as a discrete analogue of the solution to the Langevin equation.
The unique strong solution of the Langevin Equation is given by

\[ X_t = e^{ct}X_0 + \sigma e^{ct} \int_0^t e^{-cs} dB_s \]

has a normal distribution.

If \( X_0 = 0 \), then \( EX_t = 0 \), and by Itô Isometry,

\[ \text{var}(X_t) = \frac{\sigma^2}{2c}(e^{2ct} - 1) \]

\[ \text{cov}(X_t, X_s) = \frac{\sigma^2}{2c}(e^{c(t+s)} - e^{c(t-s)}), \ s < t. \]

For a constant initial condition \( X_0 \), this process is called an Ornstein-Uhlenbeck process (widely used in statistical physics and interest rate modeling).
Remark

- For any deterministic function $f(t)$, stochastic integral
  \[
  \int_0^t f(s) dB_s \sim N \left( 0, \int_0^t f(s)^2 ds \right).
  \]

- For any two Itô diffusions $X_t$ and $Y_t$, the product rule is given by
  \[
  d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t \quad \text{extra term}
  \]

- Most stochastic integrals and stochastic differential equations cannot be calculated or solved in closed forms, and so simulations must be used.