A point about the divisibility assumption for the Smalltown police LP.

Ideally we need to insist that all $x_j$'s are integers, as they model # cops. But in this case, the optimal solution will have integer values. This happens because a special property is satisfied. We will revisit this topic later.

Blending problem as LP

WV-IMP Pg 92, Prob 5:

5 Chandler Oil Company has 5,000 barrels of oil 1 and 10,000 barrels of oil 2. The company sells two products: gasoline and heating oil. Both products are produced by combining oil 1 and oil 2. The quality level of each oil is as follows: oil 1—10; oil 2—5. Gasoline must have an average quality level of at least 8, and heating oil at least 6. Demand for each product must be created by advertising. Each dollar spent advertising gasoline creates 5 barrels of demand and each spent on heating oil creates 10 barrels of demand. Gasoline is sold for $25 per barrel, heating oil for $20. Formulate an LP to help Chandler maximize profit. Assume that no oil of either type can be purchased.

Assumption about blending: the volume of product is equal to the volumes of crudes (or raw materials) mixed, i.e., there is no volume lost.
<table>
<thead>
<tr>
<th></th>
<th>Quality</th>
<th>Ad vert.</th>
<th>Price</th>
<th>Oil 1</th>
<th>Oil 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas</td>
<td>≥ 8</td>
<td>5 b/ $</td>
<td>$25/ba</td>
<td>q.ty: 10</td>
<td>5</td>
</tr>
<tr>
<td>Oil</td>
<td>≥ 6</td>
<td>10 b/ $</td>
<td>$20/ba</td>
<td>q.ty: 5000</td>
<td>10000</td>
</tr>
</tbody>
</table>

**Decisions:**
1. How much gas & heating oil to make?
2. How much to spend on ads to raise demand?

1a. How much oil1 and oil2 to be used for making gas?
1b. How much oil1 and oil2 to be used for heating oil?

**Variables:**
- $x_{ig}$: # barrels of oil i used to make gas, i=1,2.
- $x_{ih}$: # barrels of oil i used to make heating oil, i=1,2.
- $y_g$, $y_h$: $\$ spent on ads for gas/heating oil.

**Objective function (maximize profit):**

$$\max \ Z = 25 (x_{i1} + x_{i2}) + 20 (x_{i1} + x_{i2}) - (y_g + y_h) \quad \text{(net profit)}$$

*Note: We could use extra variables to model the total amount of gas and total amount of heating oil. But we still need the split variables $x_{ig}, x_{ih}, i=1,2$.)*
Constraints

Limit on availability of oils 1 and 2.

\[ x_{1g} + x_{1h} \leq 5000 \quad \text{(oil 1 limit)} \]
\[ x_{2g} + x_{2h} \leq 10000 \quad \text{(oil 2 limit)} \]

All generated demand for gasoline and heating oil must be satisfied.

\[ x_{1g} + x_{2g} \geq 5y_g \quad \text{(demand for gas)} \]

Barrels of gas demand generated by spending \( y_g \) $ on ads.

\[ x_{1h} + x_{2h} \geq 10y_h \quad \text{(demand for heating oil)} \]

Average quality of gas and heating oil must meet required standards.

\[
\frac{10x_{1g} + 5x_{2g}}{x_{1g} + x_{2g}} \geq 8 \quad \text{(quality of gas)}
\]

Average quality of gasoline - the average is taken over the volume mixed.

\[
\frac{10x_{1h} + 5x_{2h}}{x_{1h} + x_{2h}} \geq 6 \quad \text{(quality of heating oil)}
\]

You could leave the constraint as is - no need to cross multiply. Packages like AMPL will automatically simplify it.
Notice that \[
\frac{10x_1g + 5x_2g}{x_1g + x_2g} \geq 8
\] is not a nonlinear constraint! We can cross multiply to get
\[
10x_1g + 5x_2g \geq 8(x_1g + x_2g),
\]
and simplify to get
\[
2x_1g - 3x_2g \geq 0,
\]
which is indeed linear.

You need not necessarily do this step of simplification — as long as you are sure of the linearity. In fact, the software AMPL will do the simplification for you!

**Sign Restrictions**

\[
x_{ig}, x_{ih}, y_{ig}, y_{ih} \geq 0, \quad i=1,2 \quad \text{(nonnegativity)}
\]

or, just write all vars \(\geq 0\) (nonnegativity).

Here is the whole LP:

\[
\text{max } z = 25(x_{ig} + x_{2g}) + 20(x_{1h} + x_{2h}) - (y_{ig} + y_{ih}) \quad \text{(net profit)}
\]

s.t.

\[
x_{ig} + x_{1h} \leq 5000 \quad \text{(oil 1 limit)}
\]

\[
x_{ig} + x_{2h} \leq 10000 \quad \text{(oil 2 limit)}
\]

\[
x_{ig} + x_{2g} \geq 5y_{ig} \quad \text{(demand for gas)}
\]

\[
x_{1h} + x_{2h} \geq 10y_{ih} \quad \text{(demand for heating oil)}
\]

\[
(10x_{ig} + 5x_{2g})/(x_{ig} + x_{2g}) \geq 8 \quad \text{(quality of gas)}
\]

\[
(10x_{1h} + 5x_{2h})/(x_{1h} + x_{2h}) \geq 6 \quad \text{(quality of heating oil)}
\]

all vars \(\geq 0\) (nonnegativity).
Net Present Value (NPV)

Money sitting (in a bank) accrues interest.
$1$ invested today becomes $(1+r)$ one year from today, where $r$ is the interest rate (given as a fraction).

e.g., for $r = 5\%$ or $0.05$,

$$1 \text{ today} \rightarrow 1.05 \text{ in 1 year}.$$  

So, $1$ now $\equiv (1+r) 1$ year from now. Equivalently,

$$\frac{1}{1+r} \text{ now} \equiv 1 1 \text{ year from now. Hence}$$

$$\frac{1}{(1+r)^k} \equiv 1 \text{ } k \text{ years from now} \quad k \geq 0, \text{ integer}$$

The net present value (NPV) of $1$ $k$ years from now is $\frac{1}{(1+r)^k}$.

Hence we could compare different amounts multiple years into the future by comparing their NPVs. This approach is handy when comparing various investments with multiple cash flows over the years.
Two investments with varying cash flows (in thousands of dollars) are available, as shown in Table 8. At time 0, $10,000 is available for investment, and at time 1, $7,000 is available. Assuming that \( r = 0.10 \), set up an LP whose solution maximizes the NPV obtained from these investments. Graphically find the optimal solution to the LP.

<table>
<thead>
<tr>
<th>Table 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash Flow (in $ Thousands) at Time</td>
</tr>
<tr>
<td>Investment</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

\[ r = 0.1 \]

(Assume that any fraction of an investment may be purchased.)

NPV of investment 1 = \(-6 - \frac{5}{1.1} + \frac{7}{(1.1)^2} + \frac{9}{(1.1)^3}\) = $2.00

NPV of investment 2 = \(-8 - \frac{3}{1.1} + \frac{9}{(1.1)^2} + \frac{7}{(1.1)^3}\) = $1.97

You will need to use your calculator, or else MATLAB or a similar package to compute these numbers. Two decimal places of accuracy will be sufficient.

Let \( x_i \) = fraction of investment \( i \) purchased, \( i=1,2 \).

If we were allowed to buy multiple copies, rather than fractions, we would model \( x_i \) as just nonnegative real numbers.
\[
\begin{align*}
\text{max } & \quad z = 2.00x_1 + 1.97x_2 \\
\text{s.t. } & \quad x_1 \leq 1 \quad (\text{fraction invest 1}) \\
& \quad x_2 \leq 1 \quad (\text{fraction invest 2}) \\
& \quad 6x_1 + 8x_2 \leq 10 \quad (\text{money available at time 0}) \\
& \quad 5x_1 + 3x_2 \leq 7 \quad (\text{money available at time 1}) \\
& \quad x_1, x_2 \geq 0 \quad (\text{non-negativity}).
\end{align*}
\]

Solving graphically, we get \((1, 0.5)\) as the optimal solution.

Hence the optimal solution is to buy all of Investment 1 and 0.5 of Investment 2.
Inventory model

WV-IMP Pg 104, prob 1.

1 A customer requires during the next four months, respectively, 50, 65, 100, and 70 units of a commodity (no backlogging is allowed). Production costs are $5, $8, $4, and $7 per unit during these months. The storage cost from one month to the next is $2 per unit (assessed on ending inventory). It is estimated that each unit on hand at the end of month 4 could be sold for $6. Formulate an LP that will minimize the net cost incurred in meeting the demands of the next four months.

\[ s_0 = \text{initial inventory} \quad (\text{taken as 0 here}) \]

\[ s_2 \rightarrow x_1 \rightarrow s_1 \rightarrow x_2 \rightarrow s_2 \rightarrow x_3 \rightarrow s_3 \rightarrow x_4 \rightarrow s_4 \quad (\text{to be sold at } $6/\text{unit}) \]

Production costs:
- $5
- $8
- $4
- $7

Storage costs:
- $2
- $2
- $2
- $6

Selling price per item at the end

The constraints will model the "flow in = flow out" condition for each month.

More on this idea in the next lecture...