Complementary Slackness Conditions (CSC)

Conditions for optimality for max -LP

\[ \text{(P)} \quad \max \ z = \overline{c}^T \overline{x} \]
\[ \text{s.t.} \quad A \overline{x} \leq \overline{b} \]
\[ \overline{x} \geq \overline{0} \]

\[ \text{(D)} \quad \min \ w = \overline{b}^T \overline{y} \]
\[ \text{s.t.} \quad A^T \overline{y} \geq \overline{c} \]
\[ \overline{y} \geq \overline{0} \]

\( A \in \mathbb{R}^{m \times n} \). To convert (P) and (D) to standard form, we add \( \overline{z} = \begin{bmatrix} \overline{s}_1 \\ \overline{s}_2 \\ \vdots \\ \overline{s}_m \end{bmatrix} \), slack variables to (P), and \( \overline{c} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \) excess variables to (D). Let \( \overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix} \) and \( \overline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \) be feasible for (P) and (D), respectively.
\( x \) and \( y \) are optimal for (P) and (D), respectively, if and only if

\[ a_i^T y = 0 \quad \text{for} \quad i = 1, \ldots, m \]

and

\[ c_j^T x = 0 \quad \text{for} \quad j = 1, \ldots, n \]

In words, the product of a slack/excess variable and the corresponding dual variable is zero at optimality, i.e., at least one of them is zero.

Alternatively, if a constraint is nonbinding, then the corresponding variable in the complementary (or dual) problem must be 0 (in the optimal solution).

In the Farmer Jones LP, \( x_1 = 3, x_2 = 2.8, s_1 = 1.2 \) is the optimal primal solution, and \( y_1 = 0, y_2 = 10, y_3 = -1 \) is the optimal dual solution. Since \( s_1 = 1.2 > 0 \), CSCs imply that \( y_1 = 0 \) (as \( s_1 y_1 = 0 \)). Indeed, we do have \( y_1 = 0 \) at optimality.

Also, if a resource is not used fully, its shadow price is zero.

CSCs hold for any pair of (P)/(D) LPs, not just for normal LPs.
We could solve (D) instead of (P), and they use CSCs to find optimal solution for (P).

**Prob 3, pg 328**

\[
\begin{align*}
\text{max} \quad & Z = 5x_1 + 3x_2 + x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 + x_3 \leq 6 \quad y_1 \geq 0 \quad \delta_1 \\
& x_1 + 2x_2 + x_3 \leq 7 \quad y_2 \geq 0 \quad \delta_2 \\
& x_1, x_2, x_3 \geq 0 \\
& y_1, y_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} \quad & W = 6y_1 + 7y_2 \\
\text{s.t.} \quad & 2y_1 + y_2 \geq 5 \quad \epsilon_1 \\
& y_1 + 2y_2 \geq 3 \quad \epsilon_2 \\
& y_1 + y_2 \geq 1 \quad \epsilon_3 \\
& y_1, y_2 \geq 0
\end{align*}
\]

Solve (D), and use CSCs to solve (P).

Optimal solution to (D) is \( y_1 = \frac{7}{3}, y_2 = \frac{1}{3}, W^* = \frac{49}{3} \).

Correspondingly,

\[
\begin{align*}
\epsilon_1 &= 2 \cdot \frac{7}{3} + \frac{1}{3} - 5 = 0 \\
\epsilon_2 &= \frac{7}{3} + \frac{2}{3} - 3 = 0 \\
\epsilon_3 &= \frac{7}{3} + \frac{1}{3} - 1 = \frac{5}{3}
\end{align*}
\]

CSCs: \( \epsilon_2 x_3 = 0 \). Since \( \epsilon_3 = \frac{5}{3} > 0 \), \( x_3 = 0 \).

\[
\begin{align*}
y_1 \delta_1 = 0, \quad y_1 = \frac{7}{3} > 0 & \quad \Rightarrow \delta_1 = 0 \\
y_2 \delta_2 = 0, \quad y_2 = \frac{1}{3} > 0 & \quad \Rightarrow \delta_2 = 0
\end{align*}
\]

Hence in (P), we get

\[
\begin{align*}
2x_1 + x_2 + \frac{x_3}{3} &= 6 \quad (1) \\
x_1 + 2x_2 + \frac{x_3}{3} &= 7 \quad (2)
\end{align*}
\]
\[2x(2)-(1) \text{ gives } \quad 3x_2 = 8 \implies x_2 = \frac{8}{3}\]

\[\implies x_1 = 7 - 2(\frac{8}{3}) = \frac{5}{3}\]

So \(x_1 = \frac{5}{3}, x_2 = \frac{8}{3}\) is optimal for (P).

Indeed, \(z^* = 5(\frac{5}{3}) + 3(\frac{8}{3}) = \frac{49}{3} = \omega^*\), which confirms the optimality of \(x_1 = \frac{5}{3}, x_2 = \frac{8}{3}\) for (P).

Of course, the use of CSCs is more widespread than indicated by the above toy example. There are classes of optimization algorithms based on each type of optimality conditions. The ones based on CSCs start with pairs of solutions \(\bar{x}\) and \(\bar{y}\) that do not satisfy all CSCs, but may be satisfy feasibility for (P) and (D), and then progressively satisfy the CSCs. The economic interpretation is also quite important.
Integer Programming (IP)

An LP in which each variable is restricted to take only integer values is called a pure integer program, or integer program by default.

\[
\begin{align*}
\text{max} \quad & Z = 3x_1 + 2x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 4 \\
\quad & 2x_1 + x_2 \leq 6 \\
\quad & x_1, x_2 \geq 0 \\
\quad & x_1, x_2 \text{ integer} \\
\text{or} \quad & x_1, x_2 \in \mathbb{Z}
\end{align*}
\]

(default: \(x_1, x_2 \in \mathbb{R}\))

Optimal LP solution is \((\frac{8}{3}, \frac{2}{3})\), which cannot even be a candidate for IP.

Solving just the LP, and rounding the optimal solution might not work either.

The feasible set consists of 8 grid (or lattice) points that have integer coordinates within the LP feasible region. This set is not convex!
An LP in which a subset of the variables is restricted to integers is called a mixed integer program or (MIP).

If the integer variables in an IP are restricted to take only 0 or 1, then it is called a binary IP, or BIP.

If you drop the integer restrictions on an IP (or MIP or BIP), you get the LP relaxation of the original problem.

When to insist on integer variables?

Consider the two related decisions
  * Intervene militarily in Syria? YES/NO?
  * If YES, # troops to send?

Both are modeled ideally by integer variables. But 3521.3 vs 3521 or 3522 soldiers does not make much of a difference. But the first decision will indeed need to be modeled by a 0/1 variable!

We will discuss how to formulate problems as BIPs or MIPs (or IPs).
1 Coach Night is trying to choose the starting lineup for the basketball team. The team consists of seven players who have been rated (on a scale of $1 = \text{poor}$ to $3 = \text{excellent}$) according to their ball-handling, shooting, rebounding, and defensive abilities. The positions that each player is allowed to play and the player's abilities are listed in Table 9.

The five-player starting lineup must satisfy the following restrictions:

1. At least 4 members must be able to play guard, at least 2 members must be able to play forward, and at least 1 member must be able to play center.
2. The average ball-handling, shooting, and rebounding level of the starting lineup must be at least 2.
3. If player 3 starts, then player 6 cannot start.
4. If player 1 starts, then players 4 and 5 must both start.
5. Either player 2 or player 3 must start.

Given these constraints, Coach Night wants to maximize the total defensive ability of the starting team. Formulate an IP that will help him choose his starting team.

This is not a made-up toy problem! Wayne Winston, one of the authors of our textbook (used to) consult(s) for the Indiana Pacers regularly!

Decisions to make: For each player, does he start or not?

**Decision Variables:**

$$x_j = \begin{cases} 1 & \text{if player } j \text{ starts} \\ 0 & \text{otherwise} \end{cases}$$

(binary variables)
Just as we did for LP models previously, we want to write all constraints and the objective function as linear functions of the variables, some of which could be restricted to take only integer values.

Restrictions

0. \( \sum_{j=1}^{7} x_j = 5 \) (five starters)

1. \( x_1 + x_2 + x_5 + x_7 \geq 4 \) (at least 4 guards)
   \( x_3 + x_4 + x_5 + x_6 + x_7 \geq 2 \) ( \( \therefore \) 2 forwards)
   \( x_2 + x_4 + x_6 \geq 1 \) ( \( \therefore \) 1 center)

2. \( \frac{3x_1 + 2x_2 + 2x_3 + x_4 + 3x_5 + 3x_6 + 3x_7}{5} \geq 2 \) (avg. ball handling)

3. \( \frac{3x_1 + x_2 + 3x_3 + 3x_4 + 3x_5 + x_6 + 2x_7}{5} \geq 2 \) (avg. shooting)

4. \( \frac{x_1 + 3x_2 + 2x_3 + 3x_4 + 3x_5 + 2x_6 + 2x_7}{5} \geq 2 \) (avg. rebounding)

(rest of the constraints in the next lecture...