**LP duality**

**primal-dual relationships**

<table>
<thead>
<tr>
<th>max LP</th>
<th>min LP</th>
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<tbody>
<tr>
<td>$\leq$</td>
<td>$\geq 0$</td>
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<tr>
<td>$\geq$</td>
<td>$\leq 0$</td>
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<td>$=$</td>
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**variables**

| $\geq 0$ | $\geq$ |
| $\leq 0$ | $\leq$ |
| $urs$ | $urs$ |

One should not try to memorize these relationships. Instead, think in terms of normal variable $\Leftrightarrow$ normal constraint, and similarly for opposite to normal variables/constraints, and $=$ $\Leftrightarrow$ urs.

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**Motivation for studying the dual LP**

**Farmer Jones LP:**

\[
\begin{align*}
\text{max } \ z &= 30x_1 + 100x_2 \quad \text{(revenue)} \\
\text{s.t. } &x_1 + x_2 \leq 7 \quad \text{(land)} \\
&4x_1 + 10x_2 \leq 40 \quad \text{(labor hrs)} \\
&10x_1 \geq 30 \quad \text{(min corn)} \\
&x_1, x_2 \geq 0
\end{align*}
\]

$v^* = 370 \text{ at } x_1 = 3, x_2 = 2.8$

Ignore for purposes of interpretation for now.

We still work the optimal solution of the original LP — and not with the one obtained by deleting the last constraint.
Find lower and upper bounds, push them up and down, respectively, until they coincide, or are close enough.

This is a standard approach to many optimization problems – start with lower and upper bounds for the quantity you are optimizing, and tighten these bounds.

How to get lower bounds \( z_L \)?

Any feasible point gives a lower bound, e.g.,
\[
x_1 = 4, \quad x_2 = 0, \quad z = 30 \times 4 = 120 = z_L.
\]

How to get upper bound \( z_U \)?

Consider (land) x 100:
\[
100x_1 + 100x_2 \leq 700
\]

But
\[
z = 30x_1 + 100x_2 \leq 100x_1 + 100x_2 \leq 700
\]
as long as \( x_1, x_2 \geq 0 \) and the coefficients involved are all positive (30, 100, 100, 100). So \( z_U = 700 \).

The goal is to get smaller and smaller \( z_U \) values. May be the (Labor hrs) constraint could give us a smaller \( z_U \) value.
10x (Labor hrs): 40x₁ + 100x₂ ≤ 400

Again \( z = 30x₁ + 100x₂ ≤ 40x₁ + 100x₂ ≤ 400 \) \( \rightarrow \) new \( z_u \)

Better (lower) \( z_u \) values can be obtained by combining both (land) & (labor hrs) constraints with multipliers \( y₁ \) and \( y₂ \), with \( y₁ ≥ 0, \ y₂ ≥ 0. \) \( \leftarrow \) need \( ≥ 0 \), as the sense of the inequality would flip if \( y_i \) is \( ≤ 0 \).

In this case, we get \( z_u = y₁ + 40y₂ \).

But we can say \( z ≤ z_u \) only if the coefficients of \( x₁ \) and \( x₂ \) in the \( z_u \) expression can be compared correctly to the coefficients in the \( z \)-expression, i.e., we need

\[ y₁ + 4y₂ ≥ 30 \text{ and} \]
\[ y₁ + 10y₂ ≥ 100 \]

In more detail,

\[ y₁ \left( x₁ + x₂ ≤ 7 \right) + \]
\[ y₂ \left( 4x₁ + 10x₂ ≤ 40 \right) \]

\[ (y₁ + 4y₂)x₁ + (y₁ + 10y₂)x₂ ≤ 7y₁ + 40y₂ \]

Also, our goal is to find the smallest \( z_u \) that works.
Combining all these restrictions gives you the dual LP.

\[ \text{min } Z_u = 7y_1 + 10y_2 \rightarrow \text{cost} \]

\[ \text{st. } \]

\[ y_1 + 4y_2 \geq 30 \rightarrow \text{corn} \]
\[ y_1 + 10y_2 \geq 100 \rightarrow \text{wheat} \] \( \text{(D)} \)

\[ y_1, y_2 \geq 0 \]

**Economic interpretation of the Dual LP**

Consider Farmer Jones' LP, but ignore the (min corn) constraint for now.

\[ y_1 \rightarrow \text{land} \]
\[ y_2 \rightarrow \text{labour hours} \]

Suppose a firm wants to buy the farming enterprise from Jones. The firm needs to make an offer, i.e., unit price, for every acre and every labor hour that Jones has. From the firm's point of view, it makes sense to buy Jones' enterprise at the minimum cost. Thus, the firm quotes prices \( y_1 \) and \( y_2 \) for each acre and each hour of labor, respectively. The total cost for the firm is

\[ W = 7y_1 + 40y_2, \] and it tries to minimize \( W \).
Hence its objective function is
\[ \min \ w = 7y_1 + 40y_2 \] (cost)

At the same time, the offer should be attractive to Jones for the deal to fly.

If Jones has 1 acre of land and 4 hrs of labor, he can farm corn in that acre and make $30 revenue. Hence, the prices offered by the firm should be such that they at least match this revenue, i.e.,
\[ y_1 + 4y_2 \geq 30 \] (match revenue from corn)

Similarly, for wheat, we should have
\[ y_1 + 10y_2 \geq 100 \] (match revenue from wheat)

\( y_1 \) and \( y_2 \) are prices quoted by the firm, and hence must be non-negative.

Putting together all conditions gives
\[ \min \ w = 7y_1 + 40y_2 \] (total cost for firm)
\[ \text{s.t.} \ y_1 + 4y_2 \geq 30 \] (match revenue from corn)
\[ y_1 + 10y_2 \geq 100 \] (match revenue from wheat)
\[ y_1, y_2 \geq 0 \] (non-negativity)
What about the (min-corn) constraint?
We get the extra variable $y_3$ in (D) for this constraint:

$$\begin{align*}
\min \ W &= 7y_1 + 40y_2 + 30y_3 \\
\text{s.t.} &\quad y_1 + 4y_2 + 10y_3 \geq 30 \quad \text{with } y_3 \text{ included} \\
&\quad y_1 + 10y_2 \geq 100 \\
&\quad y_1, y_2 \geq 0, y_3 \leq 0
\end{align*}$$

Jones was making 30 bushels of corn per week. Hence, when the firm buys Jones' business, it will have at least 30 bushels of corn, which it could sell at a price (of $-y_3$), and make some revenue that offsets its total cost.

So, it sells each bushel of corn at $-y_3$ dollars, such that $y_3 \leq 0$.

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**Economic interpretation of the dual of a min-IP**

Leary chemicals

$$\begin{align*}
\min \ W &= 4x_1 + x_2 \quad \text{(cost)} \\
\text{s.t.} &\quad 3x_1 + x_2 \geq 10 \quad \text{(demand A)} \\
&\quad x_1 + x_2 \geq 5 \quad \text{('' B') } y_2 \\
&\quad x_1 \quad \geq 3 \quad \text{('' C') } y_3 \\
&\quad x_1, x_2 \geq 0
\end{align*}$$
\[ \begin{align*}
\text{max } w &= 10y_1 + 5y_2 + 3y_3 \\
\text{s.t. } & \\
3y_1 + y_2 + y_3 & \leq 4 \\
y_1 + y_2 & \leq 1 \\
y_1, y_2 & \geq 0
\end{align*} \] (D)

We could think of a firm offering to sell the chemicals A, B, C (i.e., the finished products) to Leary. Hence Leary has the option to just purchase the finished products from this firm, rather than produce them by running processes 1 and 2.

More details in the next lecture...