Next week: Tuesday-lecture by S. Ibrahim
Thursday-no class

HW9 is due Friday, Nov 1 by 4 PM in my mailbox

\[
\begin{align*}
\text{max } z &= 30x_1 + 25x_2 \\
\text{s.t. } &x_1 + x_2 \leq 7 \text{ } x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

The optimal solution is \( \text{A}(7,0) \).
Revenue from corn is higher than that from wheat. Also, a minimum level of corn need to be produced.
<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( e_3 )</th>
<th>( a_3 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-30</td>
<td>-25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>40</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>-M-30</td>
<td>-25</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>0</td>
<td>-3M</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>40</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

\[ I_3 \text{ under columns of } s_1, s_2, a_3 \]

\[ B^{-1} \text{ under } s_1, s_2, a_3 \]

Optimal solution is \( x_1 = 7 \), \( s_2 = 12 \), \( e_3 = 4 \), giving \( z^* = 210 \).

Basis is \( \{e_3, s_2, x_1\} \) in that order.

We can read off \( B^{-1} \) from the optimal tableau!

Idea: In \( B^{-1}N \) (under \( x_N \)), if some columns of \( N \) form \( I \), then we have \( B^{-1} \) sitting in those columns.

More generally, if some columns of \( A \) form \( I \), then \( B^{-1}A \) will have \( B^{-1} \) in those columns.
We have the identity matrix under the slack and artificial variables ($s_i$ and $a_i$).

$$B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Check: $BB^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$$C_B^T = \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 30 & 0 & 0 \end{bmatrix}.$$

Suppose coefficient of $x_2$ in the objective function changes from 25 to 25 + $\Delta$.

Questions: 1. For what range of values of $\Delta$ is the current basis optimal?

2. If, for some $\Delta$, the current basis is not optimal, how do we find the new optimal tableau quickly? Without starting from scratch and resolving the LP all over again.
The entries in the $x_2$-column are given below.

\[
\begin{align*}
\bar{a}_2 & \quad \text{is the column in A under } x_2, \\
& \quad \text{i.e., } \bar{a}_2 = \begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix}.
\end{align*}
\]

We set $c_2 = a_2^{5} + \Delta$, and evaluate the entries. Notice that only the Row-0 entry changes because of $\Delta$, rest of the entries remain the same.

\[
\begin{array}{c}
\begin{bmatrix}
1 & 0 & \bar{a}_2 \\
\end{bmatrix}
\end{array}
\begin{array}{c}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\end{array}
\begin{array}{c}
\begin{bmatrix}
1 \\
10 \\
0 \\
\end{bmatrix}
\end{array}
\begin{array}{c}
= \\
\frac{5 - \Delta}{1} \\
\frac{1}{6} \\
\end{array}
\begin{array}{c}
\text{or need } 30 \text{ here for current basis to remain optimal}
\end{array}
\]

Need $5 - \Delta \geq 0$, i.e.,

$\Delta \leq 5$ → "reduced cost" of wheat (or of $x_2$)

Notice that as long as the revenue per acre of wheat is $\leq 30$, it is still beneficial to farm only corn. But once this number is $> 30$, which happens when $\Delta > 5$, it makes sense to farm as much wheat as possible (we still need to make at least 30 bushels of corn).
**Def.** Reduced cost of a nonbasic variable in a max-LP is the maximum amount by which its objective function coefficient can increase while the current basis still remains optimal.

If the coefficient increases beyond the reduced cost, this nonbasic variable can enter the basis (current basis is no longer optimal). We can pivot this variable into the basis, and continue with the simplex method if need. For instance, consider $\Delta = 7$, for which the row-0 coefficient becomes $5 - \Delta = -2$. Hence $x_2$ can enter the basis now. We pivot $x_2$ into the basis, and get the new optimal tableau in one pivot step.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_5$</th>
<th>$s_3$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>210</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>6</td>
<td>-4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>86/3</td>
<td>1/3</td>
<td>0</td>
<td>M</td>
<td>214</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5/3</td>
<td>-1/6</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2/3</td>
<td>1/6</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5/3</td>
<td>-1/6</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

The new optimal solution is $x_1 = 5$, $x_2 = 2$, i.e., at $P(5, 2)$, where $z^* = 214$. 
Consider changing $c_i$ from 30 to $30 + \Delta$. Since an entry in $\overline{C}_B$ is changing now, more terms could be changed, as compared to when an entry in $\overline{C}_N$ is changing.

$$\overline{C}_B^{-1} = \begin{bmatrix} 0 & 0 & 30 + \Delta \\ 1 & 0 & -1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 30 + \Delta & 0 & 0 \end{bmatrix}.$$  

Current basis remains optimal as long as all $c_i \geq 0$. We just need to look at $j$ for $x_j$ nonbasic. $c_j = 0$ if $x_j$ is basic.

New $-\overline{C}_N^T + \overline{C}_B^{-1}N = \begin{bmatrix} x_2 & x_1 & a_3 \end{bmatrix} + \begin{bmatrix} 30 + \Delta & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ a_3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -25 & 0 & M \end{bmatrix} + \begin{bmatrix} 30 + \Delta & 30 + \Delta & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 + \Delta & 30 + \Delta & M \end{bmatrix} \geq 0$$

We get $5 + \Delta \geq 0$ and $30 + \Delta \geq 0$. So, $\Delta \geq -5$ i.e., $\Delta \geq -5$.

As long as revenue per acre of corn is at least $25$, which is the same as that for wheat, we continue to farm corn in all 7 acres.
(3) Changing the right-hand side (rhs) of a constraint

Consider changing \( b_i \) to \( b_i + \Delta \), where \( b_i \) is the rhs value of constraint \( i \). The rhs column \( \bar{b} \) is changed from \( \bar{b} \) to \( \bar{b} + \bar{e}_i \Delta \), where

\[
\bar{e}_i = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(\( i \)th unit m-vector)

The current basis remains optimal as long as new \( \bar{b} \geq 0 \).

Notice that the Row-0 numbers are not affected by this change.

(i) Change the # aues available from 7 to 7 + \( \Delta \)

\[
\bar{b} = \begin{bmatrix}
7 \\
40 \\
3 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
7 + \Delta \\
40 \\
3 \\
\end{bmatrix} = \begin{bmatrix}
7 \\
40 \\
3 \\
\end{bmatrix} + \Delta \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]

New rhs is \( B^{-1} \bar{b} = \begin{bmatrix}
1 & 0 & -1 \\
4 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
7 + \Delta \\
40 \\
3 \\
\end{bmatrix} = \begin{bmatrix}
4 \\
12 \\
7 \\
\end{bmatrix} + \Delta \begin{bmatrix}
1 \\
4 \\
1 \\
\end{bmatrix} \geq 0
\]

\(
\Rightarrow \quad \left\{ \begin{array}{c}
4 + \Delta \geq 0 \\
12 - 4\Delta \geq 0 \\
7 + \Delta \geq 0
\end{array} \right\} \Rightarrow \Delta \geq -4 \\
\Delta \leq 3
\Rightarrow \Delta \geq -7
\Rightarrow -4 \leq \Delta \leq 3
\)
As long as there are at least 3 ares ($\Delta = -4$), and at most 10 ares ($\Delta = 3$), we will continue to farm only corn in all of the land. If so happens that in this case, even when $b_i = 11$, say, i.e., $\Delta = 4$, we would still farm only corn. But the (land available) constraint will no longer be binding, as we have labor hours to farm corn in at most 10 ares.

We can find the shadow price of the (land available) constraint using the new optimal solution ($B^tB$), or equivalently by finding the new $z^* = c^T_b B^t \bar{b}$.

$$
\text{New } z^* = c^T_b B^t \bar{b} = \begin{bmatrix} \bar{z}_3 & \bar{z}_2 & x_1 \end{bmatrix} \begin{bmatrix} 4 + \Delta \\ 12 - 4\Delta \\ 1 + \Delta \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \Delta \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = 210 + 30\Delta.
$$

shadow price

Hence, Jones would be willing to pay up to $30 to gain access to an extra acre of land. Notice that this price is equal to the revenue from an acre of corn.