Sensitivity analysis on Farmer Jones LP

Current solution $H(3,28)$ remains optimal for $\alpha < c_1 \leq 40$ ($c_1$ is the coefficient of $x_1$ in the objective function).

Now, consider changing coefficient of $x_2$ in the objective function. To make the problem more interesting, let us assume that $c_1 = 50$ (say, price/bushel of corn is $5$).

$$\max Z = 50x_1 + 100x_2$$

s.t.

$$x_1 + x_2 \leq 7$$
$$4x_1 + 10x_2 \leq 40$$
$$10x_1 \geq 30$$
$$x_1, x_2 \geq 0$$

New $Z^* = 450$ at $F(5,2)$.

Now, say coefficient of $x_2$ in $Z$ is $c_2$.

Q: For what values of $c_2$ is the current solution optimal?
Both (labor hrs) and (land available) constraints are binding at \( F(5,2) \), the current optimal solution. \( F(5,2) \) remains optimal as long as the slope of the \( z \)-line is in between the slopes of both these constraint equations.

\[
\frac{-1}{1} \leq \frac{-50}{c_2} \leq \frac{-4}{10}
\]

\( \text{slope of } 4x_1 + 10x_2 = 40 \)

\( \text{slope of } x_1 + x_2 = 7 \)

\( \text{slope of } z = 50x_1 + c_2x_2 \)

\[ 1 \geq \frac{50}{c_2} \geq \frac{4}{10} \quad \text{(multiplying by -1)} \]

\[ \Rightarrow \quad 50 \leq c_2 \leq 125 \]

We could specify the corresponding range for the price per bushel of wheat by dividing the above range for \( c_1 \) by 25 bu/acre, which is the yield for wheat.

\[ 2 \leq \text{price} \cdot \text{wheat} \leq 5 \]
For instance, if price/bushel of wheat becomes $3.5, \( F(5, 2) \) is still optimal. But the new \( z^* \) is

\[
z^* = 50(5) + 87.5(2) = 425.
\]

So far we have not altered the feasible region. Only the slope of the \( z \)-line changes. Now we consider changing the constraint RHS values.

Sensitivity analysis for change in RHS value

Stick with \( z = 50x_1 + 100x_2 \). Suppose Jones has 45 hrs of labor. Would the current basis still remain optimal?

As long as (land) and (labor hrs) constraints are binding, the optimal basis remains the same. (Optimal solution and \( z^* \) will change).
If you get 90+ in Final, it will replace your midterm grade. If you get 80-89.9, the weights would be midterm—10%, Final—30%.

In general, let the rhs value of the labor hrs constraint be \( b_2 \). For what values of \( b_2 \) is the current basis \( (BV=\{x_1, x_2, e_3\}) \) optimal?

As long as \( 4x_1 + 10x_2 = b_2 \) is (at or) below \( G \) and (at or) above \( A \), current basis remains optimal.

At \( A(7,0) \), \( b_2 = 4 \times 7 + 10 \times 0 = 28 \), and

at \( G(3,4) \), \( b_2 = 4 \times 3 + 10 \times 4 = 52 \).

Hence, for \( 28 \leq b_2 \leq 52 \), the current basis is optimal.

What is the effect of changing labor hours on \( z^* \)?

Assume \( b_2 \) changes from 40 to \( 40 + \Delta \), within the range found above.
i.e., $28 \leq 40 + \Delta \leq 52$, which gives 

$$-12 \leq \Delta \leq 12.$$ 

The new optimal solution (new $f$) is given by solving the following system:

$$4x_1 + 10x_2 = 40 + \Delta$$
$$x_1 + x_2 = 7$$

$$6x_2 = 12 + \Delta \Rightarrow x_2 = 2 + \frac{\Delta}{6}$$

$$\Rightarrow x_1 = 5 - \frac{\Delta}{6}$$

So, the new optimal solution is at $(5 - \frac{\Delta}{6}, 2 + \frac{\Delta}{6})$.

New $z^* = 50(5 - \frac{\Delta}{6}) + 100(2 + \frac{\Delta}{6}) = 450 + 100\frac{\Delta}{6} - 50\frac{\Delta}{6}$

$$= 450 + \frac{25\Delta}{3}$$

(initial $z^*$)

Shadow price of
labor hours constraint
Def. The shadow price of constraint $i$ is the amount by which the value of $z$ improves (increases for a max-LP, or decreases for a min-LP) for a unit increase in $b_i$ (rhs value of constraint $i$), assuming the optimal basis remains the same after the change in $b_i$.

Shadow price of (land available) constraint:

\[
\begin{align*}
  x_1 + x_2 &= 7 + \Delta \\
  4x_1 + 10x_2 &= 40 \\
  6x_2 &= 12 - 4\Delta \\
  \Rightarrow x_2 &= 2 - \frac{2}{3}\Delta \\
  \Rightarrow x_1 &= 5 + \frac{5}{3}\Delta
\end{align*}
\]

New optimal solution is $(5 + \frac{5}{3}\Delta, 2 - \frac{2}{3}\Delta)$.

New $z^* = 50\left(5 + \frac{5}{3}\Delta\right) + 100\left(2 - \frac{2}{3}\Delta\right) = 450 + \frac{50}{3}\Delta$.

\[\Rightarrow\] shadow price of (land available) constraint $= \frac{50}{3}$.
Shadow price of the (min corn) constraint is zero, as $30 \rightarrow 30+\Delta$ will not change the optimal solution $x_1(5,2)$. In general, shadow price of a non-binding constraint is zero.

**Economic interpretation of shadow price**

Shadow price of (labor-hrs) constraint is $\frac{25}{3}$. This means Jones would be willing to pay up to $\frac{25}{3}$ for each extra hour of labor. Up to that unit price, he will at least break even.

Say Jones has the option of employing one more worker, and has to decide how much he can pay him per hour while still maintaining the revenue. This price (per extra hour of labor) is the shadow price in question. Hence, if he can get extra hours of labor at, say, $6$, he will indeed take up that option, as $6 < \frac{25}{3}$. 