Alternative optimal solutions in tableau simplex method

WV-IMP pg 154, Prob 3

\[
\begin{align*}
\text{max } Z &= x_1 + x_2 \\
\text{s.t. } &x_1 + x_2 + x_3 \leq 8_1 \\
&x_1 + 2x_3 \leq 8_2 \\
&x_i \geq 0, i=1,2,3
\end{align*}
\]

\[
\begin{array}{cccccccc}
\hline
Z & x_1 & x_2 & x_3 & s_1 & s_2 & \text{RHS} \\
\hline
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 2 & 0 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 2 & 0 & 1 & 1 \\
\hline
\end{array}
\]

\{ optimal bfs \} \quad \begin{align*}
X_1 &= 1, S_2 &= 0, Z^* &= 1 \equiv A \\
X_2 &= 1, S_2 &= 1, Z^* &= 1 \equiv B \\
X_1 &= 1, X_2 &= 0, Z^* &= 1 \equiv A
\end{align*}

Notice that you could pivot \( S_2 \) back into the basis in the last tableau. \( S_2 \) will replace \( X_1 \) back into the BFS corresponding to \( B \) (with \( BV = \{ z, x_2, S_2 \} \)).
In the space of input variables \((x_1, x_2, x_3)\), we have identified two distinct corner points that are optimal, namely, 

\[ A : \bar{X}_A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B : \bar{X}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} . \]

All points on \(\overline{AB}\) are optimal solutions as well.

Any point on \(\overline{AB}\) can be written as

\[
\bar{x} = \alpha A + (1-\alpha) B \quad \text{for} \quad 0 \leq \alpha \leq 1 .
\]

\[
= \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ 1-\alpha \\ 0 \end{bmatrix} , \quad \text{for} \quad 0 \leq \alpha \leq 1 .
\]

Indeed \(z^* = x_1 + x_2 = \alpha + (1-\alpha) = 1\) for any such \(\bar{x}\).

The point given as \(\bar{x} = \alpha \bar{X}_A + (1-\alpha) \bar{X}_B\), for \(0 \leq \alpha \leq 1\)

is a convex combination of \(\bar{X}_A\) and \(\bar{X}_B\). By the convexity property of the LP feasible region, any convex combination of optimal solutions is also optimal.
Idea in 3D (and higher dimensions):

The $z$-plane hits push against an entire face, here shown with five corner points $\overline{v}_j$, $j=1,5$.

Each corner point $\overline{v}_j$ is optimal, and so is any point in the shaded region. Any point in the pentagon is a convex combination of the $\overline{v}_j$s.

**Def.** A convex combination of $\overline{v}_1, \overline{v}_2, ..., \overline{v}_n$ is

$$\overline{x} = \sum_{j=1}^{n} \alpha_j \overline{v}_j, \quad 0 \leq \alpha_j \leq 1, \quad \sum_{j=1}^{n} \alpha_j = 1.$$  

For instance, when $\alpha_2 = 1$, $\alpha_j = 0$ for $j=1,3,4,5$, $\overline{x} = \overline{v}_2$. Similarly, when $\alpha_3 = \alpha_5 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \alpha_4 = 0$, we get $\overline{x} = \frac{1}{2} (\overline{v}_3 + \overline{v}_5)$, which is the midpoint of the line segment connecting $\overline{v}_3$ and $\overline{v}_5$. And when $\alpha_j = \frac{1}{5}$ for all $j$, $\overline{x}$ is the “centroid” (or average) of all the corner points.
Unbounded LPs in tableau simplex method

WV-IMP Pg 158, prob 1

\[ \begin{align*}
\text{max } z &= 2x_2 \\
\text{s.t. } x_1 - x_2 &\leq 4 \quad &s_1 \\
-x_1 + x_2 &\leq 1 \quad &s_2 \\
x_1, x_2 &\geq 0
\end{align*} \]

Recall that in 2D, when you could slide the z-line without limits while improving z and remaining feasible, the LP is unbounded.

\[
\begin{array}{cccccc}
& x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
1 & 0 & -2 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 4 \\
0 & -1 & 0 & 1 & 0 & 1 \\
\hline
1 & -2 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 5 \\
0 & -1 & 1 & 0 & 0 & 1 \\
\end{array}
\]

if this entry were -3 instead, the LP is still unbounded!

\[ z = 2 + 2x_1 \\
s_1 = 5 \\
x_2 = 1 + x_1 \]

\( x_1 \) could enter the basis, but we do not have a candidate for min ratio test. Hence the LP is unbounded.

We could increase \( x_1 \) while keeping \( s_2 \) and \( x_2 \geq 0 \), i.e., feasible, and improve z-value without limit.
In the last tableau, suppose the Row-0 coefficient of $s_2$ were -3, instead of 2. By following our default rule of choosing the nonbasic variable with the most negative Row-0 coefficient to enter the basis, we could have pivoted $s_2$ into the basis rather than consider $x_1$. At the same time, the $x_1$ column would have revealed the unboundedness of the LP irrespective of whether $s_2$ could enter the basis or not. Indeed, if we were to pivot $s_2$ into the basis, the unboundedness would become evident from a subsequent tableau.

As demonstrated here, the $x_1$-column indicates unboundedness by itself. We could conclude the LP is unbounded just based on the $x_1$-column here.

So far, we have seen how to detect unique solution, alternative optimal solutions, and unboundedness in tableau simplex. We now consider the fourth case, i.e., infeasible LP.
The Big-M Method (not covered in midterm!)

The big-M is a large positive number, which could be used in place of +\infty. But we could do meaningful arithmetic on expressions involving M—so, \( 3M+5 > M-10000 \), for instance.

WV-DMP pg 178, Prob 2

\[
\begin{align*}
\text{min } Z &= 2x_1 + 3x_2 \\
\text{s.t. } & & 2x_1 + x_2 \geq 4 \quad (1) \\
& & x_1 - x_2 \geq -1 \quad (2) \\
& & x_1, x_2 \geq 0
\end{align*}
\]

**Step 1** Modify any constraints so that all rhs values are non-negative. Recall that we can read off the bfs from the tableau—assuming all rhs values are \( \geq 0 \). Else, feasibility is violated.

If the rhs value is negative, multiply constraint by \(-1\). Notice that the sense of the inequality reverses in this process.

\[(2) \times -1 \Rightarrow -(x_1 - x_2 \geq -1) \Rightarrow -x_1 + x_2 \leq 1 \quad (2')\]

For instance, consider \(-3 \geq -5\). Multiplying this inequality by \(-1\) indeed reverses the sense of the inequality:

\[(-(3 \geq -5) \Rightarrow 3 \leq 5.\]
One advantage of using slack variables is that we can choose the obvious starting bfs by picking the slack variables in the BV. But for '≥' constraints, we subtract excess variables, which are not canonical. Similarly, we do not have obvious canonical variables for '=' constraints. Hence, we add artificial variables for such constraints.

**Step 2** Add an artificial variable \( a_i \) for constraint \( i \) if it is a '≥' or an '=' constraint, and add nonnegativity for each \( a_i \): \((a_i ≥ 0)\).

(1): \[ 2x_1 + x_2 + a_1 ≥ 4 \] \((1')\)

**Step 3** For a max LP, add \(-Ma_i\) to the objective function \( z \), and for a min LP add \(+Ma_i\), where \( M \) is a large positive number.

\[
\begin{align*}
\text{min } \quad & z = 2x_1 + 3x_2 + Ma_i \\
\text{s.t.} \quad & 2x_1 + x_2 + a_i ≥ 4 \quad (1') \\
& -x_1 + x_2 ≤ 1 \quad (2') \\
& x_1, x_2, a_i ≥ 0
\end{align*}
\]

→ this term forces \( a_i \) to zero in any optimal solution, assuming the LP is not infeasible. With the M coefficient, as long as \( a_i > 0 \), \( z \) is very huge due to the \( Ma_i \) term, however small \( a_i \) is.