Midterm on Tue, Oct 8, in SLOAN 32.
From Nov 5 onward, class to meet in CUE 316

Simplex method

\[
\begin{align*}
\text{max } & \quad Z = 2x_1 - x_2 + x_3 \\
\text{s.t. } & \quad 3x_1 + x_2 + x_3 \leq 60 \quad s_1 \\
& \quad x_1 - x_2 + 2x_3 \leq 10 \quad s_2 \\
& \quad x_1 + x_2 - x_3 \leq 20 \quad s_3 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

We can represent all the numbers in a compact table format, called the simplex tableau (pronounced "tableau"). All calculations are also efficiently represented in this format. This version of the simplex method is called the tableau simplex method.

Each tableau corresponds to a bfs, assuming it is constructed correctly. In fact, we could directly go to the starting tableau from the given LP.
The optimal solution is \( x_1 = 15, x_2 = 5, \lambda_1 = 10 \), with \( Z^* = 25 \).

Let us recall the idea of the min ratio test, explaining it on the first tableau. Here, \( BV = \{ x_1, x_2, x_3 \} \), \( NBV = \{ x_1, x_2, x_3 \} \). Increasing \( x_1 \) or \( x_3 \) (from zero) will increase the \( Z \)-value. We pick \( x_1 \) as the entering variable, as the rate of increase is higher. Thus, \( x_1 \) is the entering variable.
Our goal is to move to an adjacent bfs at which the z-value is better (larger for a max LP). To move to an adjacent bfs, we exchange one basic variable with a current nonbasic variable. Here, we are going to include \( x \) in the basis, and remove one of the current basic variables from the BV set. The min ratio test helps us to identify which variable we should remove.

The 3 constraint equations in the first tableau read as follows:

\[
\begin{align*}
2x_1 + s_1 &= 60 \quad \Rightarrow \quad s_1 = 60 - 3x_1 \\
3x_1 + s_2 &= 10 \\
x_1 + s_3 &= 20 \\
\quad s_2 = 10 - x_1 \\
\quad s_3 = 20 - x_1
\end{align*}
\]

We need to keep \( s_1 \geq 0, s_2 \geq 0, s_3 \geq 0 \) for feasibility. Hence we get \( 60 - 3x_1 \geq 0, 10 - x_1 \geq 0, 20 - x_1 \geq 0 \), or equivalently,

\[ x_1 \leq 60/3, x_1 \leq 10, x_1 \leq 20, \]

which all hold when \( x_1 \leq 10 \).

When \( x_1 > 10 \), \( s_2 \) becomes negative, i.e., we are no longer feasible. So \( x_1 = 10 \) is the winner of the min ratio test, and since this ratio comes from Row 2, in which \( s_2 \) is canonical at present, the entering variable \( x_1 \) replaces \( s_2 \) from BV set (i.e., \( s_2 \) leaves the basis).

Notice that if we had \( s_2 = 10 + x_1 \) (instead of \( - \)), then increasing \( x_1 \) would not affect the nonnegativity of \( s_2 \). This is the reason why we do not consider rows for the min ratio test that have negative (or zero) coefficients for the entering variable.
Simplex method for min LPs

The criteria for choosing the entering variable and to decide optimality of a bfs are opposite to those used for a max LP.

* Current bfs is optimal if the coefficients of all variables in Row-0 are nonpositive (≤0).
* The variable with the largest positive number in Row-0 enters the basis.
* The min-ratio test is the same as for max LP.

WV-IMP pg 151, Prob 1

\[ \begin{align*}
\text{min } Z &= 4x_1 - x_2 \\
\text{s.t. } & 2x_1 + x_2 \leq 8 \\
& x_2 \leq 5 \\
& x_1 - x_2 \leq 4 \\
& x_1, x_2 \geq 0
\end{align*} \]

\[
\begin{array}{ccccccc}
\text{z} & x_1 & x_2 & s_1 & s_2 & s_3 & \text{rhs} \\
\hline
1 & -4 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & 0 & 8 \\
0 & 0 & 1 & 0 & 1 & 0 & 5 \\
0 & 1 & 1 & 0 & 0 & 1 & 4 \\
\hline
1 & -4 & 0 & 0 & -1 & 0 & -5
\end{array}
\]

\[ Z - 4x_1 - x_2 = -5 \]

\[ \Rightarrow Z = -5 + 4x_1 + s_2 \]

all numbers ≤ 0 \(\Rightarrow\) current bfs is optimal

Optimal solution is \( x_2 = 5, s_1 = 3, s_3 = 9 \), with \( Z^* = -5 \).
\[ \text{min } Z = -x_1 - x_2 \]
\[ \text{s.t. } \]
\[ x_1 - x_2 \leq 1 \]
\[ x_1 + x_2 \leq 2 \]
\[ x_1, x_2 \geq 0 \]

An optimal solution is \( x_1 = \frac{3}{2}, x_2 = \frac{1}{2} \), with \( z^* = -2 \).

\[
\begin{array}{ccccccc}
2 & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 2 \\
\hline
1 & 0 & 2 & -1 & 0 & -1 \\
0 & 1 & -1 & 1 & 0 & 1 \\
0 & 0 & 2 & -1 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & -1 & -2 \\
0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

indicates alternative optimal solutions

\( B(\frac{3}{2}, \frac{1}{2}) \) and \( C(0, 2) \) are both optimal corner points. Any point on \( BC \) is also optimal, giving \( z^* = 2 \).
If the coefficient of a nonbasic variable in Row-0 of an optimal tableau is zero, there exist alternative optimal solutions.

\[ \bar{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}, \quad \bar{x}_C = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ are both optimal, with } z^* = -2. \]

\[ \bar{x} = \alpha \bar{x}_B + (1-\alpha) \bar{x}_C \text{ for } 0 \leq \alpha \leq 1 \text{ represents all points on } \overline{BC}. \]
Notice that we can represent the entire line segment $\overline{BC}$ by expressing a general point $P$ on the line segment as

$$P = \alpha B + (1-\alpha)C, \quad 0 \leq \alpha \leq 1.$$ 

When $\alpha=0$, we get $P=C$. Similarly, when $\alpha=1$, we get $P=B$.

When $\alpha=0.5$, we get $P = \frac{B+C}{2}$, i.e., the midpoint of $\overline{BC}$.

Hence, 

$$\overline{x} = \alpha \overline{x}_B + (1-\alpha) \overline{x}_C = \alpha \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\alpha/2 \\ 2 - 3\alpha/2 \end{bmatrix}, \quad 0 \leq \alpha \leq 1$$

represents all optimal solutions. Indeed, we have

$$z^* = -x_1 - x_2 = -\frac{3\alpha}{2} - (2 - \frac{3\alpha}{2}) = -2.$$