Prob 35 (Section 13.5) \( (D_{\hat{u}} f)_{P_0} = (\nabla f)_{P_0} \cdot \hat{u} \)

\( \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \)

Setting: You're given \( (D_{\hat{u}} f)_{P_0} \) and \( (D_{\overline{w}} f)_{P_0} \), and asked to find \( (D_{\overline{w}} f)_{P_0} \) for directions \( \overline{u} \), \( \overline{w} \), and \( \overline{v} \) (not necessarily unit vectors). Form of \( f \) is not given.

\( (D_{\overline{u}} f) \) at \( P_0(1,2) \) in direction of \( \overline{u} = \hat{i} + \hat{j} \) is \( 2\sqrt{2} \) and in direction of \( \overline{v} = -2\hat{j} \) is \(-3\). Find \( (D_{\overline{w}} f)_{P_0} \) in the direction of \( \overline{w} = -\hat{i} - 2\hat{j} \).

\( (D_{\overline{w}} f)_{P_0} = (\nabla f)_{P_0} \cdot \overline{w} = (\nabla f)_{P_0} \cdot \frac{\overline{w}}{||\overline{w}||} \)

Let \( (\nabla f)_{P_0} = (f_x)_{P_0} \hat{i} + (f_y)_{P_0} \hat{j} = f_1 \hat{i} + f_2 \hat{j} \) where \( f_1 \) and \( f_2 \) are unknown.

\( (D_{\overline{u}} f)_{P_0} = (\nabla f)_{P_0} \cdot \overline{u} = (f_1 \hat{i} + f_2 \hat{j}) \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}} = 2\sqrt{2} \)

i.e. \( \sqrt{2}(f_1 + f_2) = 2\sqrt{2} \), which gives \( f_1 + f_2 = 4 \) \( -\text{(1)} \)

\( (D_{\overline{w}} f)_{P_0} = (\nabla f)_{P_0} \cdot \overline{w} = (f_1 \hat{i} + f_2 \hat{j}) \cdot \frac{-2\hat{j}}{\sqrt{2}} = \frac{f_2 - 2}{\sqrt{2}} = -3 \)

i.e.

\( f_2 = 3 \) \( -\text{(2)} \)

So \( (\text{1}) \) gives \( f_1 = 1 \).
\[(\nabla f)_{P_0} = 4\hat{i} + 3\hat{j} = \hat{i} + 3\hat{j}\.
\]

Hence \[(\nabla f)_{P_0} \cdot \frac{\overrightarrow{w}}{||\overrightarrow{w}||}
\]

\[
= (\hat{i} + 3\hat{j}) \cdot \frac{(-\hat{i} - 2\hat{j})}{\sqrt{5}}
\]

\[
= (1 \cdot -1 \frac{1}{\sqrt{5}} + 3 \cdot -2 \frac{1}{\sqrt{5}}) = -\frac{7}{\sqrt{5}}.
\]

**Tangent Planes and Differentials (Section 13.6)**

Extend idea of tangent line of a level curve to that of the tangent plane of a level surface.

Let \( \vec{F}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \) be a smooth curve on a level surface \( f(x, y, z) = c \).

We write \( f(x(t), y(t), z(t)) = c \) and apply chain rule (w.r.t. \( t \)).

\[
\frac{df}{dt} = \frac{dc}{dt} = 0
\]

\[
\frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} + \frac{df}{dz} \cdot \frac{dz}{dt} = 0
\]

\[
(\frac{df}{dx} \hat{i} + \frac{df}{dy} \hat{j} + \frac{df}{dz} \hat{k}) \cdot (\frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}) = 0
\]

\[\nabla f \quad \quad \frac{d\vec{r}}{dt} \rightarrow \text{velocity vector}\]
Hence $\nabla f$ is orthogonal to the velocity vector.

All tangent lines to the surface (i.e., tangents to all level curves on the surface) at given point $P_0$ lie on a plane that is orthogonal to $\nabla f$ at $P_0$.

**Def.** The tangent plane at $P_0$ on the level surface $f(x,y,z) = c$ of a differentiable function $f$ is the plane through $P_0$ orthogonal to $(\nabla f)_p$.

The line parallel to $(\nabla f)_p$, passing through $P_0$, is the normal line of the surface at $P_0$.

Section 11.5: Equation of plane perpendicular to $A\hat{i} + B\hat{j} + C\hat{k}$ at $P_0(x_0,y_0,z_0)$ is $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$.

Equation of a line through $P_0$ parallel to $\overrightarrow{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is $\overrightarrow{r}(t) = \overrightarrow{r}_0 + t\overrightarrow{v}$ for $-\infty < t < \infty$, i.e.,

$$\overrightarrow{r}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \text{ i.e., } \begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$$
Hence, the tangent plane to \( f(x, y, z) = c \) at \( P_0(x_0, y_0, z_0) \) is
\[
\left( \frac{\partial f}{\partial x} \right)_{P_0} (x-x_0) + \left( \frac{\partial f}{\partial y} \right)_{P_0} (y-y_0) + \left( \frac{\partial f}{\partial z} \right)_{P_0} (z-z_0) = 0.
\]

The normal line is given by
\[
x = x_0 + \left( \frac{\partial f}{\partial x} \right)_{P_0} t, \quad y = y_0 + \left( \frac{\partial f}{\partial y} \right)_{P_0} t, \quad z = z_0 + \left( \frac{\partial f}{\partial z} \right)_{P_0} t, \quad -\infty < t < \infty.
\]

**Prob 2** Find equations of (a) tangent plane and (b) normal line at \( P_0(3, 5, -4) \) to the surface \( x^2 + y^2 - z^2 = 18 \).

\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}
\]
\[
= 2x \hat{i} + 2y \hat{j} - 2z \hat{k}
\]

at \( P_0(3, 5, -4) \),
\[
\nabla f = 6 \hat{i} + 10 \hat{j} - 8 \hat{k}
\]

So, the tangent plane is \( 6(x-3) + 10(y-5) - 8(z+4) = 0 \)

i.e., \( \frac{1}{2}(6x + 10y + 8z + (-18 - 50 + 32) = 0) \)

i.e., \( 3x + 5y + 4z = 18 \).

Normal line is \( x = 3 + 6t, \)
\[
y = 5 + 10t
\]
\[
z = -4 + 8t, \quad -\infty < t < \infty.
\]