Double Integrals in Polar Form (Section 14.4)

Recall: polar coordinates \( r, \theta \)

\((x, y) \equiv (r \cos \theta, r \sin \theta)\)

\[ r = \sqrt{x^2 + y^2} \]

\[
\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta) \, r \, dr \, d\theta
\]

In polar coordinates, \( dA = (r \, d\theta) \cdot dr = r \, dr \, d\theta \)

\[ \text{NOT} \, \, dr \, d\theta! \]
3. Describe region $R$ in polar coordinates.

\[
\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\
0 \leq r \leq \csc \theta
\]

Describe in polar coordinates

7. The region enclosed by the circle $x^2 + y^2 = 2x$.

\[
x^2 + y^2 = 2x \\
x^2 - 2x + 1 + y^2 = 1 \\
(x-1)^2 + y^2 = 1
\]

\[-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\
0 \leq r \leq 2 \cos \theta.
\]

Notice that this computation is equivalent to the ray-shooting method. In the previous example, we can do a similar computation on $y=1$.

Finding limits of integration in polar coordinates

1. Sketch region.
2. Find $r$ limits (shoot ray (arrow) from origin - find $r=g_1(\theta)$ where it enters $R$, and $r=g_2(\theta)$ where it leaves $R$).
3. Find $\theta$ limits.
11. Change the Cartesian integral to equivalent polar integral. Then evaluate the polar integral.

\[ I = \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-y^2}} (x^2+y^2) \, dx \, dy \]

\[ x = \sqrt{4-y^2}, \ x^2 + y^2 = 4 \]

\[ 0 \leq \theta \leq \frac{\pi}{2}, \ 0 \leq r \leq 2 \]

In polar form

\[ I = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{4} \sin \theta} r^2 \, r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \left( \int_0^2 r^3 \, dr \right) \, d\theta = \int_0^{\frac{\pi}{2}} \frac{r^4}{4} \bigg|_0^2 \, d\theta \]

\[ = \frac{16}{4} \int_0^{\frac{\pi}{2}} d\theta = 4 \cdot \theta \bigg|_0^{\frac{\pi}{2}} = 4 \cdot \frac{\pi}{2} = 2 \pi. \]

25. Sketch region \( R \), and convert integral to Cartesian form.

\[ I = \int_0^{\frac{\pi}{2}} \int_0^{2 \sec \theta} r^5 \sin^2 \theta \, dr \, d\theta \]

At \( \theta = 0, \ 2 \sec \theta = 2 \)

\( \theta = \frac{\pi}{4}, \ 2 \sec \theta = \frac{2}{\sqrt{2}} = 2\sqrt{2} \)

\[ 0 \leq x \leq 2 \]

\[ 0 \leq y \leq x \]
\[ I = \iint_R r^4 \sin^2 \theta \, r \, dr \, d\theta = \iint_R \frac{r^2 - r^2 \sin^2 \theta}{y^2} y^2 \, r \, dr \, d\theta \]

\[ = \iint_R (x^2 + y^2) \, y^2 \, dy \, dx = \iint_0^2 (x^2 + y^2) \, y^2 \, dy \, dx. \]

**Area in Polar Coordinates**

\[ A = \iint_R r \, dr \, d\theta = \iint_R r \, dr \, d\theta \]

29. Find area of region cut from the first quadrant by the curve \( r = 2(2 - \sin 2\theta)^{1/2} = 2\sqrt{2 - \sin 2\theta} \).

\[
\begin{array}{c|c|c}
\theta & r & \frac{r}{2} \\
\hline
0 & 2\sqrt{2} & 2 \sqrt{2} \\
\frac{\pi}{2} & 2 & 2 \\
\frac{\pi}{4} & & \\
\end{array}
\]

\[ 0 \leq \theta \leq \frac{\pi}{2} \]

\[ 0 \leq r \leq 2\sqrt{2 - \sin 2\theta} \]
\[ A = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} r \, dr \, d\theta \]

\[ = \int_{0}^{\frac{\pi}{2}} \left( \frac{1}{2} r^2 \right)_{0}^{2\sqrt{1-\sin^2 \theta}} \, d\theta \]

\[ = \frac{1}{2} \left[ 4(2-\sin 2\theta) \right]_{0}^{\frac{\pi}{2}} = 4 \theta \left[ \frac{\pi}{2} \right. + \cos 2\theta \left. \right]_{0}^{\frac{\pi}{2}} \]

\[ = 4 \left( \frac{\pi}{2} - 0 \right) + (\cos \pi - \cos 0) = 2\pi + (-1 - 1) \]

\[ = 2\pi - 2 = 2(\pi - 1) \]

We will not talk about triple integrals or polar coordinates in 3D due to time constraints. These topics extend the ideas we introduced in 2D to 3D.