7. (14) Sketch the region of integration, and write an equivalent integral with the order of integration reversed. Then evaluate this reverse ordered integral.

\[ I = \int_0^1 \int_{x^2}^x \sqrt{x} \, dy \, dx. \]

I uses vertical cross sections.

Y varies from \( x^2 \) to \( x \), and

X varies from 0 to 1.

Points of intersection of \( y = x \) and \( y = x^2 \):

\[ x = x^2, \quad \text{i.e.} \quad x(x-1) = 0, \]

Giving \( x = 0,1 \), for which \( y = 0,1 \).

The points of intersection are \((0,0)\) and \((1,1)\).

Reversing the order of integration, we write

\[ I = \int_0^1 \int_{\sqrt{y}}^{y} \sqrt{x} \, dx \, dy. \]

\[ = \int_0^1 \left[ \frac{2}{3} \sqrt{y} \right]_{\sqrt{y}}^{y} \, dy = \frac{2}{3} \int_0^1 \left( y^{3/2} - (\sqrt{y})^{3/2} \right) \, dy \]

\[ = \frac{2}{3} \left[ \frac{4}{7} y^{7/4} - \frac{2}{5} y^{5/2} \right]_0^1 = \frac{2}{3} \left[ \frac{4}{7} (1) - \frac{2}{5} (1) \right] - 0 \]

\[ = \frac{2}{3} \left( \frac{4 \times 5 - 2 \times 7}{7 \times 5} \right) = \frac{2 \times 6}{3 \times 35} = \frac{4}{35}. \]
6. (12) Evaluate the double integral over the given region \( R \).

\[
I = \iint_R xy e^{xy^2} \, dA, \quad R : 0 \leq x \leq 2, \ 0 \leq y \leq 1.
\]

\[
I = \int_0^2 \int_0^1 xy e^{xy^2} \, dy \, dx
\]

\[
\begin{align*}
&= \int_0^2 \left( \frac{1}{2} e^{xy^2} \right)_0^1 \, dx \\
&= \frac{1}{2} \int_0^2 \left( e^x - 1 \right) \, dx \\
&= \frac{1}{2} \left[ e^x - x \right]_0^2 \\
&= \frac{1}{2} \left[ e^2 - 2 - (e^0) \right] \\
&= \frac{1}{2} \left( e^2 - 3 \right).
\end{align*}
\]

Notice that

\[
\frac{\partial}{\partial y} (e^{xy^2}) = e^{xy^2} \cdot x = x(2y) \cdot e^{xy^2} = 2xy e^{xy^2}
\]

so, integrating first w.r.t. \( x \) is much harder here!

8. (6) Decide whether each of the following statements is True or False. Justify your answer.

(a) A point that gives the absolute maximum of a function in a given region \( R \) must also be a local maximum of the function.

(b) Swapping the lower and upper limits of both integrals in a double integral leaves the value of the double integral unchanged.

(a) FALSE. The absolute maximum could occur on the boundary of \( R \).

(b) TRUE. Each swap multiplies the integral by \(-1\), so the value is unchanged as \((-1)(-1) = 1\).
3. (12) Let \( y = uv \). If \( u \) is measured with an error of 2\% and \( v \) is measured with an error of 3\%, estimate the percentage error in the calculated value of \( y \).

\[
y = uv
\]

The total differential of \( y \) is

\[
\frac{1}{y} (dy = u dv + v du).
\]

We want \( \frac{dy}{y} \), given

\[
\frac{du}{u} = 2\%, \quad \frac{dv}{v} = 3\%.
\]

Equivalently, \( \frac{du}{u} \times 100 = 2 \), \( \frac{dv}{v} \times 100 = 3 \).

\[
\frac{dy}{y} = \frac{udv}{uv} + \frac{vdv}{uv} = \frac{dv}{v} + \frac{du}{u} = 2\% + 3\% = 5\%.
\]

y = uv gives

\[
\frac{dy}{y} = \frac{udv}{uv} + \frac{vdv}{uv} = \frac{dv}{v} + \frac{du}{u} = 2\% + 3\% = 5\%.
\]

5. (16) Find the absolute maximum and minimum values of \( f(x, y) = x^2 + xy + y^2 - 3x + 3y \)
on the region \( R \) that is the part of the line \( x + y = 4 \) lying in the first quadrant.

\( R \) cannot have interior critical points.

\( R \) is \( \overline{AB} \) from \( A(4,0) \) to \( B(0,4) \).

On \( \overline{AB} \), \( y = 4 - x \), hence

\[
f(x, 4-x) = f(x) = x^2 + x(4-x) + (4-x)^2 - 3x + 3(4-x)
= x^2 + 4x - x^2 + x^2 - 8x + 16 - 3x + 12 - 3x
= x^2 - 10x + 28
\]

\[
f'(x) = 2x - 10 = 0 \text{ gives } x = 5, \text{ giving } y = 4 - 5 = -1.
\]

But \((5, -1)\) is not on \( \overline{AB} \). So we just check \( f(x, y) \)
at \( A(4,0) \) and \( B(0,4) \).
\[ f(x, y) = x^2 + xy + y^2 - 3x + 3y \]

A: \( f(4, 0) = (4)^2 + 0 + 0 - 3(4) + 0 = 4 \leftarrow \text{absolute minimum} \)

B: \( f(0, 4) = (0)^2 + 0 + (4)^2 - 0 + 3(4) = 28 \leftarrow \text{absolute maximum} \)

4. (14) Find all local minima, local maxima, and saddle points of the function given below. You should evaluate the function at each critical point.

\[ f(x, y) = x^3 + y^3 - 3xy + 15. \]

The domain is all of \( \mathbb{R}^2 \) (all real pairs).

Critical points

\[ f_x = 3x^2 - 3y = 0 \quad (1) \]

\[ f_y = 3y^2 - 3x = 0 \quad (2) \]

(2) \( \Rightarrow \) \( x = y^2 \). Plugging into (1) gives

\[ 3(y^2)^2 - 3y = 0 \quad \Rightarrow \quad y^4 - y = 0 \quad y(y^3 - 1) = 0, \]

giving \( y = 0, 1 \), and hence \( x = 0, 1 \). So the critical points are \((0, 0)\) and \((1, 1)\).

\[ f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3; \quad \text{So} \]

\[ H = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9. \]

\[ (0, 0) \]

\[ H = 36(0)(0) - 9 = -9 < 0 \]

\( \Rightarrow \) \((0, 0)\) is a saddle point.

\[ f(0, 0) = 15. \]

\((0, 0, 15)\) is a saddle point.

\[ (1, 1) \]

\[ H = 36(1)(1) - 9 = 27 > 0. \]

\[ f_{xx} = 6(1) = 6 > 0. \quad \Rightarrow \text{\((1, 1)\) is a local minimum.} \]

\[ f(1, 1) = (1)^3 + (1)^3 - 3(1)(1) + 15 = 14. \]

\((1, 1, 14)\) is a local minimum.