11. Write an integral for \( \iint_R \, dA \) over region \( R \) using 
(a) vertical cross sections, and (b) horizontal cross section.

(a) \[ \int_0^3 \int_0^{3x} \, dy \, dx \]

(b) \[ \int_0^{\sqrt{9}} \int_0^{\sqrt[3]{y}} \, dx \, dy \]

12. Sketch the region of integration, and evaluate the integral.

\[ \int_0^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy \]

Plot \( x = \ln y \), i.e., \( y = e^x \).

The right point of intersection has \( x = \ln (\ln 8) = \ln \ln 8 \).
\[ \int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy = \int_0^{\ln 8} e^y \left( \int_0^{\ln y} e^x \, dx \right) \, dy = \int_0^{\ln 8} (e^{y+x} \big|_0^{\ln y}) \, dy \]

\[ = \int_1^{\ln 8} (e^{y+\ln y} - e^{y+0}) \, dy = \int_1^{\ln 8} (e^{y+\ln y} - e^y) \, dy \]

\[ = \int_1^{\ln 8} (ye^y - ey) \, dy = \left[ ye^y - ey \big|_1^{\ln 8} \right] \]

\[ \frac{d}{dy}(ye^y - ey) = ye^y \]

\[ = (\ln 8 e^{\ln 8} - 2e^{\ln 8}) - (e^{\ln 8} - 2e^{\ln 8}) \]

\[ = 8\ln 8 - 16 + e = e + 8(\ln 8 - 2). \]

Let's evaluate the integral now by reversing the order of integration.

\[ \int_0^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy = \int_0^{\ln 8} (e^{x+y} \big|_0^{\ln y}) \, dx \]

\[ = \int_1^{\ln 8} \left( e^{x+\ln 8} - e^{x+e^x} \right) \, dx \]

\[ = \int_0^{\ln 8} \left( 8e^x - e^x e^x \right) \, dx \]
\[ \int_0^{\ln 8} \left( 8e^x - e^x e^x \right) dx = 8e^x - e^x \bigg|_0^{\ln 8} \]

Recall:
\[ \frac{d}{dx} (e^{f(x)}) = e^{f(x)} f'(x) \]

\[ = \left( 8e^{\ln 8} - e^{\ln 8} e^{\ln 8} \right) - \left( 8e^0 - e^0 \right) \\
= 8 \times 8 - 8 - 8 + 1 = e + 8(\ln 8 - 2). \]

47. Sketch region of integration, reverse the order of integration and evaluate the integral.

\[ I = \int_0^\pi \int_x^\pi \frac{\sin y}{y} dy \\ x dy \\
originally, vertical cross sections are used. we reverse to use horizontal cross sections.

\[ I = \int_0^\pi \int_0^y \frac{\sin y}{y} dx dy \\
= \int_0^\pi \left( \frac{\sin y}{y} \times \left|_0^y \right. \right) dy \\
= \int_0^\pi \left( \frac{\sin y}{y} (y - 0) \right) dy \\
= \int_0^\pi \sin y dy \\
= -\cos y \bigg|_0^\pi = 1 - 1 = 2. \]
In all the integrals we have seen so far, the region of integration $R$ is bounded essentially by two curves. Notice that even in the case where $R$ is a triangle, as seen in the example above, $y = x$ and $y = \pi$ were sufficient to describe it, along with $x = 0$. Now we consider more general regions $R$, which we split into component regions $R_1, R_2, R_3, \text{etc.}$, where each component region is simpler, just as we have seen so far.

55. Find $I = \iint_R (y - 2x^2) \, dA$ where $R$ is the region bounded by the square $|x| + |y| = 1$.

$|x| + |y| = 1$ splits into four lines:

$\pm x \pm y = 1$, i.e.,

$x + y = 1$

$x - y = 1$

$-x + y = 1$

$-x - y = 1$

The region is bounded by 4 curves, instead of 2. So split into two regions bounded by two curves each.

... we'll finish this problem in the next lecture...