Double integration over general domains \(^{(14.2)}\)

**Theorem 2** (Fubini’s stronger theorem)

Let \(f(x,y)\) be a continuous function on region \(R\).

1. \(\) If \(R\) is defined by \(a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\), with \(g_1(x)\) and \(g_2(x)\) are continuous over \(x \in [a, b]\), then

\[
\iint_{R} f(x,y) \, dA = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx.
\]

2. \(\) If \(R\) is defined by \(c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\), with \(h_1(y)\) and \(h_2(y)\) are continuous over \(y \in [c, d]\), then

\[
\iint_{R} f(x,y) \, dA = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy.
\]

For a given integral \(\iint_{R} f(x,y) \, dA\), we could use either of these two forms, and we should get the same answer.

We first try to evaluate such a double integral, and then provide details of how to specify the details of the region of integration.
19. Sketch the region of integration \( R \) and evaluate the integral \( \iint_R x \sin y \, dy \, dx \).

The integral is of the form described in Option 1, \( \int_a^b g_2(x) \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx \), with \( g_1(x) = 0 \) and \( g_2(x) = x \). Or, the limits of \( y \) are \( y = 0 \) to \( y = x \), and limits of \( x \) are \([0, \pi]\).

More generally, one needs to plot \( g_1(x) \) and \( g_2(x) \), and decide which sides of these two curves to pick.

\[
\iint_R x \sin y \, dy \, dx = \int_0^\pi \int_0^x x \sin y \, dy \, dx = \int_0^\pi \left( x \left( -\cos y \right) \right|_0^x \, dx
\]

\[
= \int_0^\pi x \left[ -\cos x - (-\cos 0) \right] \, dx = \int_0^\pi x (1 - \cos x) \, dx
\]

\[
= \left[ \left( x - x \cos x \right) \right|_0^\pi = \left[ \frac{1}{2} x^2 - (x \sin x + \cos x) \right|_0^\pi
\]

\[
= \frac{1}{2} (\pi^2) - \pi \sin \pi - \cos \pi - \left( \frac{0^2}{2} + 0 \sin 0 + \cos 0 \right) = \frac{\pi^2}{2} + 2.
\]
Sketching Regions of Integration

Procedure using vertical cross sections

1. Sketch region and label bounding curves.
2. Imagine a vertical line crossing the region at $x$, and figure out the limits of $y$ as functions of $x$.
3. Find the limits for $x$, such that the region includes all possible vertical lines as used in Step 2.

The procedure using horizontal cross sections is similar, except that the roles of $x$ and $y$ are reversed.

Sketch the region of integration $0 \leq x \leq 3, 0 \leq y \leq 2x$.

Since the limits of $y$ are given as functions of $x$ here, we are indeed using vertical cross sections. But notice that we could equivalently describe the region as

$0 \leq y \leq 6, \ \frac{y}{2} \leq x \leq 3$;

using horizontal cross sections.
3. \(-2 \leq y \leq 2, y \leq x \leq 4\)

\[ x = y^2 \text{ gives } y = \sqrt{x} \]

\[ x = y^2 \] has the shape of the parabola \( y = x^2 \), but with \( x \) and \( y \) flipped.

(a). Write the integral for \( \iint_R \, dA \) over region \( R \) using

(a) vertical cross sections and (b) horizontal cross sections.

\[ 0 \leq x \leq 3 \sqrt[3]{y} \]

\[ y = 8 \]

\[ y = x^3 \]

\[ x = 2 \]

Notice that for the vertical line cutting across the region, \( y \) varies from \( x^3 \) to 8.

Also, \( y = x^3 \) and \( y = 8 \) intersect at \((2, 8)\).

For the horizontal line crossing the region, \( x \) varies from 0 to \((y)^{\frac{1}{3}}\), i.e., \( 3 \sqrt[3]{y} \).