Calculus III - Section 2

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See the syllabus on the course web page. All info and documents relevant for this course will be available from the webpage. In particular, no paper versions will be handed out in class.

Functions of several variables  (Section 13.1 from the book).

**Def** Let D be a set of n-tuples of real numbers \((x_1, \ldots, x_n)\).

A real-valued function \(f\) on D is a rule that assigns a unique real number

\[
W = f(x_1, \ldots, x_n)
\]

to each element in D.

D is the domain of \(f\). The set of real values \(W\) taken by \(f\) is its range.

\(f\) is a function of the independent variables \(x_1, \ldots, x_n\).

\(W\) is the dependent variable.

Another set of related terminology calls \(x_1, \ldots, x_n\) the input variables, and \(W\) the output variable.

One could think of "putting in" \(x_1, x_2, \ldots, x_n\) into the "box" that is \(f\), and getting out \(W\). For a given tuple \((x_1, \ldots, x_n)\), the output \(W\) is unique.
Notation

In 2D, we write \( z = f(x, y) \).
In 3D, we write \( w = f(x, y, z) \).

Indeed, we will “draw” the 2D function in 3D using \((x, y, z)\) coordinates!

Evaluating functions

\[ f(x, y) = \sin(xy) \]

(b) \[ f(-3, \frac{\pi}{2}) = \sin\left(-3 \times \frac{\pi}{2}\right) = \sin\left(-\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}. \]

Domains and Ranges (of functions)

We extend the ideas used to define the domain of functions of a single variable to those of several variables. As in the 1D case, we look to avoid division by zero and negative values inside square roots.

**Def.** The domain of \( f \) is the largest set of variable values that generate real numbers as output values.

The range of \( f \) is the set of all real output values.

Notice that both the domain and the range are sets.
Prob 5 (pg 692)

\( f(x,y) = \sqrt{y-x-2} \). Find and describe the domain of \( f \).

We need \( y-x-2 \geq 0 \), as the number inside the square root must be nonnegative. Indeed, the domain \( D \) of \( f \) here is the set of all pairs \((x,y)\) of values in real 2D space that satisfy \( y-x-2 \geq 0 \).

Let's try to visualize \( D \). For sets described by an inequality of this sort, we first plot the equation, and then pick the correct side of the line.

We first plot \( y-x-2=0 \), or \( y=x+2 \). This is the 45° line passing through \((0,2)\) and \((-2,0)\).

To pick the correct side, we try any point on the inequality to see if it holds. If it holds, that point is on the correct side. Since we could try any point, we try \((0,0)\) to keep things simple. But, \(0-0-2 \neq 0\). So \((0,0)\) is on the wrong side.
We now define properties of sets or regions that could describe the domain (and the function itself). In particular, we want to know if $D$ is "open" or "closed", "bounded" (or finite) or "unbounded" etc.

**Def** A point $P(x,y)$ in a set $R$ in the plane is an **interior point** of $R$ if $P$ is the center of a disc of some positive radius lying entirely in $R$.

$P(x,y)$ is a **boundary point** of $R$ if every disc centered at $P$ has some points inside $R$ and some outside. Notice that $P$ itself need not be in $R$.

The set of all interior points: interior of $R$. The set of all boundary points: boundary of $R$.

If $R$ contains only interior points, it is **open**. If $R$ contains its boundary, it is **closed**.

The set $R$ here is bounded.

$R$ here is open, as it does not contain boundary points.