Estimating changes in a specific direction

In 1D, change in $f(x)$ at $x = p_0$ is estimated by
\[ df = f'(p_0) \, dx \] for small increment $dx$

derivative x increment

differential of $f$ at $x = p_0$

Extending to higher dimensions,
\[ df = (\nabla f \cdot \hat{u}) \, ds, \] where $ds$ is the change in the
directional derivative x increment in the direction of $\hat{u}$

Problem: By about how much will $g(x, y, z) = x + x \cos z - y \sin z + y$ change when $p(x, y, z)$ moves from $P_0 (2, -1, 0)$ toward the
point $P_1 (0, 1, 2)$ a distance of $ds = 0.2$ units?

\[
\nabla g = \frac{\partial g}{\partial x} \hat{i} + \frac{\partial g}{\partial y} \hat{j} + \frac{\partial g}{\partial z} \hat{k}
\]
\[
= (1 + \cos z - 0 + 0) \hat{i} + (0 + 0 - \sin z + 1) \hat{j} + (0 - x \sin z - y \cos z + 0) \hat{k}
\]
\[
= (1 + \cos z) \hat{i} + (1 - \sin z) \hat{j} - (x \sin z + y \cos z) \hat{k}
\]
\[
(\nabla g)_{p_0} = (1 + \cos 0) \hat{i} + (1 - \sin 0) \hat{j} - (2 \sin 0 + 0 \cos 0) \hat{k}
\]
\[
= 2 \hat{i} + \hat{j} + \hat{k}.
\]
Direction $\vec{u} = \vec{P_0P_1} = (0-2)\hat{i} + (1-(-1))\hat{j} + (2-0)\hat{k}$

$P_0(2, -1, 0)$
$P_1(0, 1, 2)$

$||\vec{u}|| = \sqrt{(2^2 + 1^2 + 2^2)} = 2\sqrt{3}$.

So, $\hat{u} = \frac{\vec{u}}{||\vec{u}||} = \frac{1}{2\sqrt{3}} (-2\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{1}{\sqrt{3}} (-\hat{i} + \hat{j} + \hat{k})$.

$\begin{align*}
(D_u g)_{P_0} &= (\nabla g)_{P_0} \cdot \hat{u} \\
&= (2\hat{i} + \hat{j} + \hat{k}) \cdot \frac{1}{\sqrt{3}} (-\hat{i} + \hat{j} + \hat{k}) \\
&= \frac{1}{\sqrt{3}} (2 \cdot -1 + 1 \cdot 1 + 1 \cdot 1) = 0.
\end{align*}$

So $dg = (D_u g)_{P_0} \cdot ds = 0 \cdot (0, 2) = 0$. 
Review for Exam 1

- domain, range, level curves, open/closed, bounded/unbounded
- partial derivatives
- chain rule
- branch diagrams
- gradient
- directional derivative

Practice Exam

8. True/False

(a) False. Take \( y = x^2 \) is closed, as it includes its boundary \( y = x^2 \). But it is unbounded.

(b) False. We draw two branch diagrams, one for each independent variable.

(c) True. Follows from properties of \( \nabla f \).

(d) False. \( (\nabla \hat{u} f) = \nabla f \cdot \hat{u} = (\nabla f \cdot |\hat{u}| \cdot \cos \theta \right)

\( |\nabla f| = 0 \) then \( (\nabla \hat{u} f) = 0 \) in all directions \( \hat{u} \).
2(a). \[ f(x, y) = \frac{x + y}{xy - 1} \]

\[
\frac{\partial f}{\partial x} = \frac{(xy-1)(1+0) - (x+y)(y-o)}{(xy-1)^2} = \frac{xy-1 - xy - y^2}{(xy-1)^2} = \frac{-y^2+1}{(xy-1)^2}
\]

\[ f(x, y) \text{ is symmetric w.r.t. } x \text{ and } y, \text{ so} \]

\[
\frac{\partial f}{\partial y} = \frac{-(x^2+y)}{(y-1)^2} = -\frac{(x^2+y)}{(xy-1)^2}. \quad f(y, x) = \frac{y+x}{y-1} = \frac{x+y}{xy-1} = f(x, y)
\]

Alternatively, we could evaluate \( \frac{\partial f}{\partial y} \) directly:

\[
\frac{\partial f}{\partial y} = \frac{(xy-1)(0+1) - (x+y)(y-0)}{(xy-1)^2} = \frac{(xy-1) - x^2 - xy}{(xy-1)^2}
\]

\[
= \frac{-(x^2+y)}{(xy-1)^2}.
\]