Introduction to Linear Algebra (Math 220_2) – Fall 2013
Practice Final

• There are twelve problems and eight pages in this exam.
• Show all work.
• Provide appropriate justifications where required.
• Good luck!

1. (6) Let \( T(x_1, x_2) = (3x_1 + 2x_2, x_1, -x_1 + 4x_2) \) be a linear transformation.
   (a) Is \( T \) one-to-one? Justify your answer.
   (b) Is \( T \) onto? Justify your answer.

2. (8) Let \( A = \begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 6 & 2 & 6 & 0 \end{bmatrix} \). Then \( \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).
   (a) Determine a basis for \( \text{Col}(A) \).
   (b) Determine a basis for \( \text{Nul}(A) \).
   (c) What is \( \text{dim}(\text{Nul}(A)) \)? Explain.
   (d) What is \( \text{rank}(A) \)? Explain.

3. (10) Let \( A \) and \( B \) be \( n \times n \) matrices. We say that \( A \) and \( B \) are similar if there is an invertible matrix \( P \) such that \( B = P^{-1}AP \). Show that if \( A \) and \( B^T \) are similar, then \( A \) and \( B \) have the same eigenvalues.

4. (10) Let \( A + B \) and \( C \) be \( n \times n \) invertible matrices. Solve the following equation for \( X \). Justify each step in your solution.
   \[ C^{-1}(XB +XA)C = C^T. \]

5. (8) The matrix \( A = \begin{bmatrix} -1 & 3 & 3 \\ -3 & 5 & 3 \\ 3 & -3 & -1 \end{bmatrix} \) has eigenvalues 2, 2 and \(-1\). Determine a basis for the eigenspace corresponding to the eigenvalue \( \lambda = 2 \).

6. (9) Let \( A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 4 \\ -1 & 2 & -1 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \).
   (a) If \( A \) is invertible, find \( A^{-1} \).
   (b) If the inverse exists, use \( A^{-1} \) computed above to solve the system \( Ax = b \).
7. (7) Construct a nonzero $3 \times 3$ matrix $A$ with rank 2, and a vector $b$ that is not in Nul $A$.

8. (8) Let $\det A = 3$ and $\det B = 2$. Evaluate each of the following quantities, if possible. Justify your answers.
   (a) $\det A^2$
   (b) $\det(2AB^T)$
   (c) $\det A^{-1}/\det B^{-1}$
   (d) $\det(A + B)$

9. (7) It is known that $x = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of a $3 \times 3$ matrix $A$ corresponding to the eigenvalue $\lambda = 0$. Is the linear transformation $T(x) = Ax$ one-to-one? Justify your answer.

10. (8) Let $A = \begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of $k$, if any, will make $AB = BA$?

11. (8) Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.
   (a) Is $\lambda = 1$ an eigenvalue of $A$? If yes, find an associated eigenvector.
   (b) Is $\lambda = -2$ an eigenvalue of $A$? If yes, find an associated eigenvector.
   (c) Is $x = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ an eigenvector of $A$? If yes, find the corresponding eigenvalue.
   (d) Is $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector of $A$? If yes, find the corresponding eigenvalue.

12. (10) Decide whether each of the following statements is True or False. Justify your answer.
   (a) If $Ax = b$ is inconsistent for some $b \in \mathbb{R}^n$, then $\lambda = 0$ is an eigenvalue of $A$.
   (b) It could happen that $\det(A + B) = \det A + \det B$.
   (c) If $x$ is an eigenvector of the matrix $A$ corresponding to the eigenvalue $\lambda$, then $3x$ is an eigenvector corresponding to the eigenvalue $3\lambda$.
   (d) If $A$ is a $3 \times 4$ matrix, the largest value that $\dim \text{Nul} A$ can take is 3.
   (e) If the system $Ax = b$ has more than one solution, then so does the system $Ax = 0$. 