A note on companion pencils

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Abstract. Various generalizations of companion matrices to companion pencils are presented. Companion matrices link to monic polynomials, whereas companion pencils do not require monicity of the corresponding polynomial. In the classical companion pencil case \((A, B)\) only the coefficient of the highest degree appears in \(B\)'s lower right corner. We will show, however, that all coefficients of the polynomial can be distributed over both \(A\) and \(B\) creating additional flexibility.

Companion matrices admit a so-called Fiedler factorization into essentially \(2 \times 2\) matrices. These Fiedler factors can be reordered without affecting the eigenvalues (which equal the polynomial roots) of the assembled matrix. We will propose a generalization of this factorization and the reshuffling property for companion pencils.

Special examples of the factorizations and extensions to matrix polynomials and product eigenvalue problems are included.

1. Introduction

It is well known that the polynomial

\[
p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0
\]

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equals the determinant \( \det(zB - A) \), with

\[
A = \begin{bmatrix}
0 & -c_0 \\
1 & 0 & -c_1 \\
& \ddots & \ddots & \ddots \\
& & 1 & 0 & -c_{n-2} \\
& & & 1 & -c_{n-1}
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 \\
& \ddots \\
& & 1 \\
& & & \ddots \\
& & & & 1
\end{bmatrix}.
\]

The matrix pencil \((A, B)\) is called the \textit{companion pencil} of \(p(z)\) and can be used to compute the roots of \(p(z)\) as the roots of \(p(z)\) coincide with the eigenvalues of the pencil \((A, B)\).

At the moment, generalizing Frobenius companion matrices and/or pencils \([3, 7, 9]\) is an active research topic as new matrix forms and/or factorizations open up possibilities to develop new, possibly faster or more accurate algorithms for both the classical rootfinding problem as well as matrix polynomial eigenvalue problems \([6, 8, 12]\). Even though the companion pencil exhibits a favorable numerical behavior \([10]\) with respect to the companion matrix, there are only few structured QZ algorithms for companion pencils available \([4, 5]\).

The backward error bound for polynomial rootfinding based on companion matrices is

\[
\|c - \tilde{c}\| < k_1 \frac{c_{\max}}{c_n} \|c\| \epsilon_m + O(\epsilon_m^2),
\]

where \(\tilde{c}\) the coefficients of a polynomial with the computed roots, \(k_2\) a constant, \(c_{\max}\) the coefficient with the largest absolute value, and \(\epsilon_m\) the machine precision. However, the backward error for polynomial rootfinding based on companion pencils is only

\[
\|c - \tilde{c}\| < k_2\|c\| \epsilon_m,
\]

where a scaling to \(\|c\| = 1\) is possible \([8, 10, 12]\). Thus polynomial rootfinding with companion pencils is advantageous for polynomials with large \(\frac{c_{\max}}{c_n}\).

In this article no new rootfinders will be proposed, only theoretical extensions of the existing companion pencil are introduced. First, in Section 2 we will generalize the companion pencil \((1)\) by distributing not only the highest order, but all polynomial coefficients over both \(A\) and \(B\). Moreover, one does not have to choose whether to put a specific coefficient \(c_i\) in \(A\) or \(B\), one can also write \(c_i = v_i + w_i\), where the term \(v_i\) will appear in \(A\), and \(w_i\) in \(B\). For matrix polynomials a related approach has been investigated in \([11]\). Second, we will adjust Fiedler’s factorization of a companion matrix to obtain a similar factorization of the companion pencil and we will prove that also in this case one can reorder all the factors without altering the pencil’s eigenvalues. Finally, in Sections 4 and 5 we link some of the results to matrix polynomials and product eigenvalue problems.

2. Generalized companion pencils

The following theorem shows that the coefficients can be distributed over the last column of \(A\) and the last column of \(B\).
Theorem 2.1. Consider \( p(z) \), a polynomial with complex coefficients \( p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \). Let \( v, w \) be vectors of length \( n \) with

\[
\begin{align*}
v_1 &= c_0, \\
v_{i+1} + w_i &= c_i, & \text{for } i = 1, \ldots, n - 1, \text{ and} \\
w_n &= c_n.
\end{align*}
\]

Then the determinant \( \det(zB - A) \) of the pencil \( (A, B) \), with

\[
A = Z - v e_n^T, \quad B = Z^H Z + w e_n^T,
\]

\( e_n \) the \( n \)th standard unit vector, and \( Z \) the downshift matrix

\[
Z = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 1 \end{bmatrix},
\]

equals \( p(z) \).

**Proof.** We have

\[
\det(zB - A) = \det \begin{pmatrix} z & zw_1 + v_1 \\ -1 & z & zw_2 + v_2 \\ & \ddots & \ddots & \ddots \\ & -1 & z & zw_{n-1} + v_{n-1} \\ -1 & & & -1 & zw_n + v_n \end{pmatrix}.
\]

Applying Laplace’s formula to the first row yields

\[
\det(zB - A) = z \det \begin{pmatrix} z & zw_2 + v_2 \\ -1 & z & zw_3 + v_3 \\ & \ddots & \ddots & \ddots \\ & -1 & z & zw_{n-1} + v_{n-1} \\ -1 & & & -1 & zw_n + v_n \end{pmatrix} + (-1)^{n+1} (zw_1 + v_1) \det \begin{pmatrix} -1 & z \\ -1 & z \\ & \ddots & \ddots \\ & -1 & z \end{pmatrix}.
\]

The proof is completed by applying Laplace’s formula recursively. \( \square \)

The coefficients \( c_0 \) in \( A \) and \( c_n \) in \( B \) are fixed. All other coefficients can be freely distributed between parts in \( A \) and parts in \( B \) as long as \( v_{i+1} + w_i \) sum up to \( c_i \). The following example demonstrates that the additional freedom can be used to create additional structure in \( A \) and \( B \).
Example 2.2. Let \( p(z) = 5z^3 - z^2 + 2z - 1 \). Then both \( A \) and \( B \) can be chosen to consist solely of non-negative entries, e.g.,

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 5 \end{bmatrix}.
\]

Another option is to choose always either \( w_i = 0 \) or \( v_{i+1} = 0 \):

Corollary 2.3. Consider again \( p(z) \) from (1). Take two disjunct subsets \( I \subset \{0, \ldots, n-1\} \) and \( J \subset \{1, \ldots, n\} \), \( I \cap J = \emptyset \) such that their union \( I \cup J = \{0, \ldots, n\} \).

Define two vectors \( v \) and \( w \) as

\[
v_{i+1} = \begin{cases} c_i & \text{if } i \in I, \\ 0 & \text{if } i \notin I, \end{cases} \quad \text{and} \quad w_j = \begin{cases} c_j & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}
\]

Then the eigenvalues of the pencil \((A, B)\) with

\[
A = Z - ve_n^T, \quad B = Z^H Z + w e_n^T,
\]

and \( Z \) be the downshift matrix coincide with the polynomial roots of \( p(z) \).

Example 2.4. Let \( p(z) = c_5 z^5 + c_4 z^4 + \cdots + c_1 z + c_0 \). Then \((A, B)\), with

\[
A = \begin{bmatrix} 0 & -c_0 \\ 1 & 0 & 0 \\ 1 & 0 & -c_2 \\ 1 & 0 & 0 \\ 1 & -c_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & c_1 \\ 1 & 0 \\ 1 & c_3 \\ 1 & 0 \\ c_5 \end{bmatrix}
\]

is a companion pencil of \( p(z) \).

Remark 2.5. When considering the existing QZ-algorithms by Boito, Eidelman, and Gemignani \cite{4,5} we notice that the algorithms are based on the property that both \( A \) and \( B \) in (2) are of unitary-plus-rank-one form, a structure maintained under QZ-iterates. This property also holds for the new pencil matrices in Theorem 2.1 implying that with modest modifications the existing algorithms could compute the eigenvalues of the new pencils as well.

3. Fiedler companion factorization

The matrix \( A \) of the pencil \((A, B)\) is the companion matrix of the monic polynomial whose coefficients are defined by \( v \). As shown by Fiedler \cite{9}, companion matrices can be factored in \( n \) core transformations. A core transformation \( X_i \) is a modified identity matrix, where a \( 2 \times 2 \) submatrix, for \( i = 0 \) and \( i = n \) one entry, on the diagonal is replaced by an arbitrary matrix; for all core transformations the subindex \( i \) links to the submatrix \( X_i(i : i + 1, i : i + 1) \) differing from the identity. For the factorization of the companion matrix the core transformations \( A_0, A_1, \ldots, A_{n-1} \), typically named Fiedler factors, have the form \footnote{Note that \( A_0 \)'s active part is restricted to the upper left matrix entry. Later \( B_n \)'s active part is restricted to the lower right matrix entry.} \( A_0(1,1) = -v_1 \).
and \( A_{i-1}(i-1 : i, i-1 : i) = \begin{bmatrix} 0 & 1 \\ 1 & v_i \end{bmatrix} \) for \( i = 2, \ldots, n \). For example, the factorization of \( A \), with \( n = 5 \) equals

\[
A = \begin{bmatrix}
0 & -v_1 \\
1 & 0 & -v_2 \\
1 & 0 & -v_3 \\
1 & 0 & -v_4 \\
1 & -v_5
\end{bmatrix} = \begin{bmatrix}
-v_1 & 0 & 1 \\
1 & -v_2 & 0 \\
1 & -v_3 & 0 \\
1 & -v_4 & 0 \\
1
\end{bmatrix} = A_0 A_1 A_2 A_3 A_4,
\]

where only the essential parts of the core transformations are shown. Moreover, Fiedler also proved [9] that reordering the Fiedler factors arbitrarily has no effect on the eigenvalues of their product matrix. Thus for all permutations \( \sigma \) of \( \{0, \ldots, n-1\} \) the eigenvalues of \( A_{\sigma(0)} \cdots A_{\sigma(n-1)} \) equal those of \( A_0 \cdots A_{n-1} \). The product \( A_{\sigma(0)} \cdots A_{\sigma(n-1)} \) is often referred to as a Fiedler companion matrix. In the setting of companion pencils Fiedler factorizations have been used, e.g., in [1, 2].

A similar factorization exists for \( B \), but since \( B \) is upper triangular two sequences of core transformations are required. For \( n = 5 \) we have

\[
B = \begin{bmatrix}
w_4 & w_3 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
w_2 & w_1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
w_1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix} = B_5 B_4 B_3 B_2 B_1 G_1 G_2 G_3 G_4,
\]

with \( B_i(i : i+1, i : i+1) = \begin{bmatrix} w_i & 1 \\ 1 & 0 \end{bmatrix} \) and \( G_i(i : i+1, i : i+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

We will now generalize Fiedler’s reordering identity of the companion matrix’ factorization to the companion pencil case.

**Theorem 3.1.** Let \((A, B)\) be a companion pencil of \( p(z) \), with \( A \) and \( B \) factored in core transformations \( A = A_0 \cdots A_{n-1} \) and \( B = B_n \cdots B_1 G_1 \cdots G_{n-1} \) as in Theorem 2.1. Then, for every permutation \( \sigma : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\} \) the pencil \((A_{\sigma}, B_{\sigma})\), with

\[
A_{\sigma} = A_0 A_{\sigma(1)} \cdots A_{\sigma(n-1)},
B_{\sigma} = B_n B_{\sigma(n-1)} \cdots B_{\sigma(1)} G_{\sigma(1)} \cdots G_{\sigma(n-1)}
\]

satisfies \( p(z) = \det(z B_{\sigma} - A_{\sigma}) \).

**Proof.** We will form \( F_{\sigma} = A_{\sigma} B_{\sigma(1)}^{-1} \cdots B_{\sigma(n-1)}^{-1} = A_{\sigma} B_{\sigma}^{-1} B_{n} \) and show that \( F_{\sigma} \) is a Fiedler companion matrix determined by \( \sigma \) and related to the monic polynomial \( \tilde{p}(z) = z^n + c_{n-1} z^{n-1} + \cdots + c_0 \), missing coefficient \( c_n \). Note that all the \( B_i \) for \( i < n \) are invertible as is \( B_n \), as otherwise the original polynomial would have had a leading coefficient 0.\(^2\) The resulting pencil \((F_{\sigma}, B_{n})\) is equivalent to the companion pencil (2) as one can prove by using the similarity transformations described in [9, Lemma 2.2] and altering the reasoning slightly not to touch transformation \( F_{n-1} \). The Fiedler companion matrix \( F_{\sigma} \) can be factored into core transformations, the Fiedler factors, \( F_0 F_{\sigma(1)} \cdots F_{\sigma(n-1)} \), with \( F_0(1, 1) = -c_0 \) and \( F_i(i : i+1, i : i+1) = \)

\(^2\)In fact, \( B_n \) does not need to be inverted, so the proof still holds even for singular \( B_n \).
The matrix $F_{n-1}$ does not commute with $B_n$. Consequently the necessary transformations involve only $F_1, \ldots, F_{n-2}$.

To construct $F_\sigma = A_\sigma B_\sigma^{-1} B_\sigma$ we will multiply the pencil $(A_\sigma, B_\sigma)$ on the right with factors to eliminate parts in $B_\sigma$ and to move them to $A_\sigma$'s side. Core transformations $X_i$ and $X_j$ commute if $|i-j| > 1$, so we can reorder $A_\sigma = A_0 A_{\sigma(1)} \cdots A_{\sigma(n-1)}$ as $A_\sigma = A_\delta = A_0 (A_{j_1-1} A_{j_1-2} \cdots A_{j_1}) \cdots (A_{n-1} A_{n-2} \cdots A_{j_r})$. We define $j_0 = 1$. This reordering appeared also in [9], and contains factors $(A_{j_{r+1}-1} A_{j_{r+1}-2} \cdots A_{j_r} A_{j_1})$ in which each core transformation $A_{j_1-1}$ is positioned to the right of $A_{j_1}$, where $j = j_\ell + 1 - 1, j_\ell + 1 - 2, \ldots, j_\ell + 1$. The ordering between these larger factors is, however, different $A_{j_1-1}$, is positioned to the left of $A_{j_1}$, with $\ell = 0, \ldots, s$.

We can also reorder the factors $B_i$ and $G_i$ to get

$$B_\sigma = B_\delta = B_0 (B_{j_1} \cdots B_{n-1}) \cdots (B_1 \cdot B_{j_1-1}) (G_{j_1-1} \cdots G_1) \cdots (G_{n-1} \cdots G_{j_s}),$$

Let us now form the product of the two inner factors

$$S = (B_1 \cdots B_{j_1-1}) (G_{j_1-1} \cdots G_1) = \begin{bmatrix} 1 & w_1 & \cdots & w_{j_1-1} \\ & 1 \\ & & \ddots \\ & & & 1 \end{bmatrix}.$$

Let $r$ be the largest index with $j_r = j_1 + r - 1$, which is in fact the number of factors $(A_{j_{r+1}-1} A_{j_{r+1}-2} \cdots A_{j_r})$ containing only a single matrix. This means that the matrix $A_\delta$ actually equals

$$A_\delta = A_0 (A_{j_1-1} A_{j_1-2} \cdots A_{j_1}) A_{j_1} \cdots A_{j_{r-1}} (A_{j_{r+1}-1} \cdots A_{j_r}) \cdots (A_{n-1} \cdots A_{j_s}),$$

where there are now $r$ factors consisting of a single matrix.

One can show that $S$ commutes with $(G_{j_2-1} \cdots G_{j_1+1}) \cdots (G_{n-1} \cdots G_{j_s})$ but not with $G_{j_1}, \ldots, G_{j_r}$. These factors $G_{j_1}, \ldots, G_{j_s}$, when applied from the right on $S$, will each move $w_{j_1-1}$ one column to the right. The matrix

$$\tilde{S} = ((G_{j_2-1} \cdots G_{j_1+1}) \cdots (G_{n-1} \cdots G_{j_s}))^{-1} S (G_{j_2-1} \cdots G_{j_1+1}) \cdots (G_{n-1} \cdots G_{j_s}),$$

is an elimination matrix. Thus inverting $\tilde{S}$ is equivalent to changing all $w_i$ into $-w_i$. By multiplying the pencil with $\tilde{S}^{-1}$ from the right, we can change it to

$$A_\delta \tilde{S}^{-1}, B_0 (B_{j_1} \cdots B_{n-1}) \cdots (B_1 \cdots B_{j_1-1}) (G_{j_1-1} \cdots G_1) \cdots (G_{n-1} \cdots G_{j_s}).$$

Again one can show that $\tilde{S}^{-1}$ commutes with $(A_{j_2-1} \cdots A_{j_1+1}) \cdots (A_{n-1} \cdots A_{j_s})$.

Further $A_{j_1}, \ldots, A_{j_s}$ will each move $w_{j_1-1}$ one column back to the left, so that

$$(A_{j_2-1} \cdots A_{j_1+1}) \cdots (A_{n-1} \cdots A_{j_s}) \tilde{S} ((A_{j_2-1} \cdots A_{j_1+1}) \cdots (A_{n-1} \cdots A_{j_s}))^{-1} = S.$$  

We can now combine $(A_{j_1-1} A_{j_1-2} \cdots A_{j_1}) S^{-1}$, yielding

$$(A_{j_1-1} \cdots A_1) S^{-1} = \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = (F_{j_1-1} \cdots F_1).$$

By repeating this procedure the remaining factors in $B_\delta$ can be moved to the left and $F_\sigma = A_0 (F_{j_1-1} F_{j_1-2} \cdots F_1) (F_{j_2-1} F_{j_2-2} \cdots F_{j_1}) \cdots (F_{n-1} F_{n-2} \cdots F_{j_s})$. □
Remark 3.2. We would like to emphasize that in Fiedler factorizations it is possible to reposition the $A_0$ term, so that it can appear everywhere in the factorization. In the pencil case, however, this is in general not possible. Nor $A_0$ nor $B_n$ can be moved freely, unless in some specific configurations, of which the Fiedler factorization with all $B_i$’s identities is one.

Let us now look at some examples of different shapes for companion pencils.

Example 3.3. Let $p(z) = 2z^4 + (\sin(t) + \cos(t))z^3 + (3+i)z^2 - 4z - 2$. A standard Hessenberg-upper triangular pencil $(A, B)$ could look like

$$A = A_0 A_1 A_2 A_3 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 0 \\ 1 & -\sin(t) \end{bmatrix}$$ and

$$B = B_1 B_2 B_3 B_4 = \begin{bmatrix} 1 & 1 \\ 1 & i \\ 1 & \cos(t) \\ 2 \end{bmatrix}.$$

Another option would be a CMV-like ordering

$$A = A_0 A_2 A_1 A_3 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 5 \\ 1 & -\sin(t) \end{bmatrix}$$ and

$$B = B_2 B_1 B_3 B_4 = \begin{bmatrix} 1 & i \\ 1 & 0 \\ 1 & \cos(t) \\ 2 \end{bmatrix}.$$

The advantage is that $A$ is pentadiagonal and $B$ the product of a pentadiagonal and a permutation matrix, properties that can be exploited when developing QR or QZ algorithms.

Example 3.4. In this example we will demonstrate the effect of the permutation $\sigma$. Let $p(z) = c_8 z^8 + \cdots + c_1 z + c_0$ and $\sigma = [1 2 3 4 5 6 7]$. Then we have

$$A_0 A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(7)}$$

$$B_8 B_{\sigma(7)} B_{\sigma(6)} \cdots B_{\sigma(1)}$$

$$G_{\sigma(1)} \cdots G_{\sigma(7)}$$

where only the active parts of the Fiedler factors are shown. We see that the shape is the same in $A$ and $G$ but mirrored in $B$. In the stacking notation above, one should replace each active matrix part by a full matrix; the notation is unambiguous as two blocks positioned on top of each other commute so their ordering does not play any role.
Remark 3.5. We focused on a specific factorization linked to the $B$-matrix. However, the matrix $B$ could also be factored as (e.g., let $n = 5$)

$$
B = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & w_1 & 0 & 1 \\
1 & 0 & w_2 & 0 & 1 \\
1 & 0 & w_3 & 0 & 1 \\
1 & 0 & w_4 & 0 & 1 \\
1 & 0 & w_5 & 0 & 1 \\
\end{bmatrix}.
$$

The factorization differs, but results similar to Theorem 3.1 can be proved.

4. Factoring companion pencils and product eigenvalue problems

Even though Theorem 2.1 is probably the most useful in practice, it is still only a special case of a general product setting, where the polynomial rootfinding problem is written as a product eigenvalue problem [13].

Theorem 4.1. Consider $p(z)$, a polynomial with complex coefficients $p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$. Define $A$, $B$, and $F^{(k)}$ (for $k = 1, \ldots, m$) based on the $n$-tuples $v, w,$ and $f^{(k)}$

$$
v_1 = c_0, \quad v_i = 0, \quad \text{for } i = 2, \ldots, n,
$$

$$
w_n = c_n, \quad w_i = 0, \quad \text{for } i = 1, \ldots, n - 1,
$$

$$
f^{(k)}_n = 0, \quad \text{and } \sum_{k=1}^m f^{(k)}_i = c_i, \quad \text{for } i = 1, \ldots, n - 1,
$$

as follows

$$
A = Z - ve_n^T = \begin{bmatrix}
0 & \cdots & -c_0 \\
1 & 0 & \cdots \\
\vdots & \ddots & \vdots \\
1 & 0 & \cdots \\
\end{bmatrix}, \quad B = Z^H Z + we_n^T = \begin{bmatrix}
1 & \cdots \\
\vdots & \ddots \\
1 & \cdots \\
\end{bmatrix},
$$

and

$$
F^{(k)} = I - f^{(k)} e_n^T = \begin{bmatrix}
1 & -f^{(k)}_1 \\
1 & \cdots \\
\vdots & \ddots \\
1 & -f^{(k)}_{n-1} \\
1 & -f^{(k)}_n \\
\end{bmatrix},
$$

with $Z$ the downshift matrix. Then the product $A(\prod_{k=1}^m F^{(k)})B_n^{-1}$ equals the companion matrix in (2).

Proof. Straightforward multiplication of the elimination matrices provides the result. The factors $F^{(k)}$ commute and thus the product notation is unambiguous. □

Theorem 2.1 is a simple consequence of Theorem 4.1. The inverse of $F^{(k)}$ is again an elimination matrix, where only a sign change of the elements determined by $f^{(k)}$ needs to be affected; as a result, we have for all $1 \leq \ell \leq m$ that the pencil $(A \prod_{k=1}^\ell F^{(k)}, B_n^{-1} \prod_{k=\ell+1}^m (F^{(k)})^{-1})$ shares the eigenvalues of the companion matrix.
matrix. Moreover, instead of explicitly computing the product one could also use the factorization to retrieve the eigenvalues of the pencil.

**Remark 4.2.** Each of the factors $F^{(k)}$ can be written in terms of his Fiedler factorization as proposed in Section 3. Moreover, Theorem 3.1 can be generalized to the product setting, where each of the factors $F^{(k)}$ obeys the ordering imposed by the permutation $\sigma$. The proof proceeds similarly to Theorem 3.1; instead of having to move only one matrix $B$ from the right to the left entry in the pencil, one has to move now each of the factors, one by one to the left.

### 5. Matrix Polynomials

In this section we will generalize the results for classical polynomials to matrix polynomials. Let $p(z) = C_n z^n + C_{n-1} z^{n-1} + \cdots + C_1 z + C_0$ be a matrix polynomial having coefficients $C_i \in \mathbb{C}^{m \times m}$. Then $p(z)$ matches the determinant $\det (zB - A)$, with

$$
A = \begin{bmatrix}
0_m & I_m & & -V_1 \\
I_m & 0_m & & -V_2 \\
& & \ddots & \vdots \\
& & & I_m - V_n
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
I_m & W_1 \\
& \ddots & \ddots \\
& & I_m & W_{n-1} \\
& & & I_m
\end{bmatrix},
$$

where $V_{i+1} + W_i = C_i$, $V_1 = C_0$, $W_n = C_n$, $I_m$ equals the identity matrix of size $m \times m$, and $0_m$ stands for the zero matrix of size $m \times m$. In the following corollary it is shown that the additional freedom in the splitting can be used to form block-matrices $A$ and $B$ with structured blocks.

**Corollary 5.1.** Let $p(z) = C_n z^n + C_{n-1} z^{n-1} + \cdots + C_1 z + C_0$ be a matrix polynomial with coefficients $C_i \in \mathbb{C}^{m \times m}$. Let further $C_0$ be skew-Hermitian ($C_0^H = -C_0$) and $C_n$ Hermitian ($C_n = C_n^H$). Then there are $V = [V_1, V_2, \ldots, V_n]^T$ and $W = [W_1, W_2, \ldots, W_n]^T$ as in (5), so that all $V_i$ are skew-Hermitian and all $W_i$ are Hermitian.

**Proof.** $V_i$ is skew-Hermitian and $W_n$ is Hermitian. For all other coefficients $C_i$ we can use the splitting $V_{i+1} = \frac{1}{2} (C_i - C_i^H)$ and $W_i = \frac{1}{2} (C_i + C_i^H)$.

In Section 4 we already showed that we can distribute the coefficients over many more factors to obtain a product eigenvalue problem. If $C_0$ and $C_n$ are of special form, one can even obtain a matrix product with structured matrices.

**Corollary 5.2.** Let $p(z) = C_n z^n + C_{n-1} z^{n-1} + \cdots + C_1 z + C_0$ be a matrix polynomial with coefficients $C_i \in \mathbb{C}^{m \times m}$, $C_0$ unitary-plus-rank-1, and $C_n = I_m$.

Then there exist $m$ upper-triangular and unitary-plus-rank-1 matrices $F^{(k)}$ and a unitary-plus-rank-1 matrix $A$, so that the eigenvalues of the polynomial coincide with those of the product eigenvalue problem $A(\prod_{k=1}^{m} F^{(k)})B_n^{-1}$.

**Proof.** Adapting the notation from Theorem 4.1 to fit matrix polynomials as in (5) and letting $V_i$ and $W_i$ as in (5) and $F^{(k)}$ as in Theorem 4.1 be matrices. The desired property holds for the $m(n-1)$ matrices $F^{(k)}_i$, with $(k = 1, \ldots, m$;  

The proof proceeds similarly to Theorem 4.1; instead of having to move only one matrix $B$ from the right to the left entry in the pencil, one has to move now each of the factors, one by one to the left.
\[ i = 1, \ldots, n - 1; \text{ and } s \text{ and } t \text{ general indices} \]

\[ F_i^{(k)}(s, t) = \begin{cases} 
C_i(s, t), & t = k, \\
0, & t \neq k, 
\end{cases} \]

as every \( F^{(k)} \) takes one column out of the coefficient matrices. \( \square \)

6. Conclusions & Future work

A generalization of Fiedler’s factorization of companion matrices to companion pencils was presented. It was shown that the pencil approach can be seen as a specific case of a product eigenvalue problem, and it was noted that all results are applicable to the matrix polynomial case.

Forthcoming investigations focus on exploiting these splittings in rootsolvers, based on companion pencils to fastly obtain reliable solutions.

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References
