1. Let $S$ be a compact subset of $C(K)$, where $K$ is a compact metric space. Prove that $S$ is equicontinuous by two different methods.

   (a) Contrapositive using sequences:
      i. Show that if $S$ is not equicontinuous, then $S$ contains a sequence $\{f_n\}$ that is not equicontinuous and has no equicontinuous subsequence.
      ii. Deduce that $S$ is not compact.

   (b) Direct proof using open covers: Hint: For each real $\epsilon > 0$ the set $\{B_{\epsilon/3}(f) \mid f \in S\}$ is an open cover of $S$.
      (Note: $B_{\epsilon/3}(f) = \{g \in C(K) \mid \|g - f\| < \epsilon/3\}$ is an open ball in $C(K)$.)

2. (Fat Cantor sets)
   Let $t$ be any real number satisfying $0 < t < 1/3$. Let $F_0 = [0, 1]$.
   Let $F_1$ be the closed set consisting of two closed intervals obtained by removing an open interval $U_{1,1}$ of length $t$ from the middle of $F_0$. Next obtain $F_2$ by removing open intervals $U_{2,1}$ and $U_{2,2}$ of length $t^2$ from the middle of each of the two closed intervals that constitute $F_1$. $F_2$ is a closed set consisting of four disjoint closed intervals. Continuing in this manner we obtain $F_3$ consisting of eight intervals, and so on. In general $F_n$ is a union of $2^n$ disjoint closed intervals. To obtain $F_{n+1}$ from $F_n$, we remove open intervals $U_{n,1}, \ldots, U_{n,k}$ ($k = 2^n$) of length $t^{n+1}$, one from the middle of each of the $2^n$ intervals that constitute $F_n$. Let

   $$F = \bigcap_{n=0}^{\infty} F_n.$$ 

   This is an uncountable compact set that contains no intervals. It is like the standard Cantor set in these respects. An important difference is that this set has positive measure, as we will see. In fact $F$ depends on the choice of $t$, so let us reset the notation and write $F_t$ instead of $F$ to indicate this dependence.

   (a) Let $G_t = [0, 1] \setminus F_t$. Compute $m(G_t)$ (Lebesgue measure) and show that for all $t \in (0, 1/3)$, $m(G_t) < 1$. Deduce that $m(F_t) > 0$. 

(b) Show that we can make \( m(F_t) \) as close to 1 as we please by choosing \( t \) appropriately.

3. **(A subset of \( \mathbb{R} \) that is not Lebesgue measurable)**

Define an equivalence relation on \( \mathbb{R} \) by \( r_1 \sim r_2 \) iff \( r_1 - r_2 \in \mathbb{Q} \). It is a simple matter to check that this is in fact an equivalence relation and therefore partitions \( \mathbb{R} \) into equivalence classes. For each \( r \in \mathbb{R} \), the equivalence class containing \( r \) is the set \( \mathbb{Q} + r = \{ q + r \mid q \in \mathbb{Q} \} \).

Let \( S \) be a set consisting of exactly one number from each equivalence class. Since each equivalence class \( \mathbb{Q} + r \) is dense in \( \mathbb{R} \), we can choose the members of \( S \) in such a way that \( S \subseteq [-1, 1] \). I claim that \( S \) is not Lebesgue measurable. We will prove this by contradiction, so assume that \( S \) is Lebesgue measurable. Let

\[
E = \bigcup \{ S + q \mid q \in \mathbb{Q} \cap [-2, 2] \}.
\]

(a) Prove that \([-1, 1] \subseteq E \subseteq [-3, 3]\).
(b) Show that if \( q_1, q_2 \in \mathbb{Q} \) and \( q_1 \neq q_2 \), then \( S + q_1 \) and \( S + q_2 \) are disjoint.
(c) Use the properties of Lebesgue measure to obtain a contradiction.

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1. In the language of abstract algebra, these equivalence classes are the *cosets* of the subgroup \( \mathbb{Q} \) in the abelian group \( \mathbb{R} \).
2. We just used the *axiom of choice*. 