The exponential function, defined by
\[
e^{z} \equiv e^{x} \cos y + i e^{x} \sin y
\]
is also written \(e^{z}\) or \(e^z\).

The exponential function is an important and useful function. We begin with one of the most important analytic functions: the complex exponential.

We have seen that the real exponential function is defined by
\[
e^{x} = e^{x} = e^{x}
\]
and that it is a real function of a real variable. The complex exponential function is defined by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

The complex exponential function is an analytic function of a complex variable.

Theorem. The complex exponential function is an analytic function of a complex variable.

Proof. The complex exponential function is defined by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Corollary. The complex exponential function is an entire function.

Proof. The complex exponential function is defined by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 1. Find the zeros of the complex exponential function.

Solution. The zeros of the complex exponential function are given by
\[
e^{z} = 0
\]
which implies
\[
e^{x} \cos y + i e^{x} \sin y = 0
\]
and
\[
e^{x} \cos y = 0
\]
and
\[
e^{x} \sin y = 0
\]
which gives
\[
x = n \pi
\]
and
\[
y = (2n + 1) \pi
\]
for \(n = 0, 1, 2, \ldots\).

Example 2. Find the first derivative of the complex exponential function.

Solution. The first derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 3. Find the second derivative of the complex exponential function.

Solution. The second derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 4. Find the third derivative of the complex exponential function.

Solution. The third derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 5. Find the fourth derivative of the complex exponential function.

Solution. The fourth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 6. Find the fifth derivative of the complex exponential function.

Solution. The fifth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 7. Find the sixth derivative of the complex exponential function.

Solution. The sixth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 8. Find the seventh derivative of the complex exponential function.

Solution. The seventh derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 9. Find the eighth derivative of the complex exponential function.

Solution. The eighth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 10. Find the ninth derivative of the complex exponential function.

Solution. The ninth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 11. Find the tenth derivative of the complex exponential function.

Solution. The tenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 12. Find the eleventh derivative of the complex exponential function.

Solution. The eleventh derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 13. Find the twelfth derivative of the complex exponential function.

Solution. The twelfth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 14. Find the thirteenth derivative of the complex exponential function.

Solution. The thirteenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 15. Find the fourteenth derivative of the complex exponential function.

Solution. The fourteenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 16. Find the fifteenth derivative of the complex exponential function.

Solution. The fifteenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 17. Find the sixteenth derivative of the complex exponential function.

Solution. The sixteenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 18. Find the seventeenth derivative of the complex exponential function.

Solution. The seventeenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 19. Find the eighteenth derivative of the complex exponential function.

Solution. The eighteenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 20. Find the nineteenth derivative of the complex exponential function.

Solution. The nineteenth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 21. Find the twentieth derivative of the complex exponential function.

Solution. The twentieth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 22. Find the twenty-first derivative of the complex exponential function.

Solution. The twenty-first derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 23. Find the twenty-second derivative of the complex exponential function.

Solution. The twenty-second derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 24. Find the twenty-third derivative of the complex exponential function.

Solution. The twenty-third derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 25. Find the twenty-fourth derivative of the complex exponential function.

Solution. The twenty-fourth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 26. Find the twenty-fifth derivative of the complex exponential function.

Solution. The twenty-fifth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 27. Find the twenty-sixth derivative of the complex exponential function.

Solution. The twenty-sixth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 28. Find the twenty-seventh derivative of the complex exponential function.

Solution. The twenty-seventh derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 29. Find the twenty-eighth derivative of the complex exponential function.

Solution. The twenty-eighth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 30. Find the twenty-ninth derivative of the complex exponential function.

Solution. The twenty-ninth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.

Example 31. Find the thirtieth derivative of the complex exponential function.

Solution. The thirtieth derivative of the complex exponential function is given by
\[
e^{z} = e^{x} \cos y + i e^{x} \sin y
\]
and is a complex function of a complex variable.