Uniqueness Results for Solutions of (1) Wave equation and (2) Heat equation

(Reference - T. Amarnath. An Elementary Course in Partial Differential Equations.)

Part A: Uniqueness of solution for one dimensional wave equation with finite length

**Theorem:** The solution of the following problem, if it exists, is unique.

\[ u_{tt} - c^2 u_{xx} = F(x,t), \quad 0 < x < l, \quad t > 0 \]  \hspace{1cm} (1)

\[ u(x,0) = f(x), \quad 0 \leq x \leq l, \]

\[ u_t(x,0) = g(x), \quad 0 \leq x \leq l, \]

\[ u(0,t) = u(l,t) = 0, \quad t > 0 \]

**Proof** The above uniqueness result for IBP of wave equation is equivalent to showing that the following IBP has only trivial solution,

\[ v_{tt} = c^2 v_{xx}, \quad 0 < x < l, \quad t > 0 \]  \hspace{1cm} (2)

\[ v(x,0) = 0, \quad 0 \leq x \leq l, \]

\[ v_t(x,0) = 0, \quad 0 \leq x \leq l, \]

\[ v(0,t) = v(l,t) = 0, \quad t > 0 \]

Let \( v(x,t) \) be a solution of problem (2). Now consider,

\[ E(t) = \frac{1}{2} \int_0^l (c^2 v_x^2 + v_t^2) dx. \]

Observe that \( E(t) \) is a differentiable function of \( t \), since \( v(x,t) \) is twice differentiable. Therefore

\[ \frac{dE}{dt} = \int_0^l (c^2 v_x v_{xt} + v_t v_{tt}) dx, \]

\[ = \int_0^l v_t v_{tt} dx + \left[ c^2 v_x v_t \right]_0^l - \int_0^l c^2 v_t v_{xx} dx. \]

\( v(0,t) = 0 \Rightarrow v_t(0,t) = 0 \quad \forall \ t \geq 0, \) and \( v(l,t) = 0 \Rightarrow v_t(l,t) = 0 \quad \forall \ t \geq 0. \)

Therefore

\[ \frac{dE}{dt} = \int_0^l v_t (v_{tt} - c^2 v_{xx}) dx = 0 \Rightarrow E = \text{constant}. \]

Since \( v(x,0) = 0 \) implies \( v_x(x,0) = 0 \) and given that \( v_t(x,0) = 0 \), therefore

\[ E(0) = 0 \Rightarrow E \equiv 0. \]
Hence \( v_x \equiv 0, v_t \equiv 0 \ \forall \ t > 0, \ 0 < x < l. \) This is possible only if \( v(x, t) = \)constant, since \( v(x, 0) = 0, \ v \equiv 0. \) Hence the theorem.

**Part B: Uniqueness of solution for one dimensional heat equation with finite length**

**Theorem:** The solution of the following problem, if it exists, is unique.

\[
\begin{align*}
\frac{u_t}{t} - \kappa \frac{u_{xx}}{x} &= F(x, t), \quad 0 < x < l, \ t > 0 \\
u(x, 0) &= f(x), \quad 0 \leq x \leq l, \\
u(0, t) &= u(l, t) = 0, \ t \geq 0
\end{align*}
\]

**Proof** The above uniqueness result for IBP of heat equation is equivalent to showing that the following IBP has only trivial solution,

\[
\begin{align*}
v_t &= \kappa v_{xx}, \quad 0 < x < l, \ t > 0 \\
v(x, 0) &= 0, \quad 0 \leq x \leq l, \\
v(0, t) &= v(l, t) = 0, \ t \geq 0
\end{align*}
\]

Let \( v(x, t) \) be a solution of problem (4). Now consider,

\[
E(t) = \frac{1}{2\kappa} \int_0^1 v^2(x, t)dx.
\]

Observe that \( E(t) \) is a differentiable function of \( t, \) since \( v(x, t) \) is twice differentiable.

Therefore

\[
\frac{dE}{dt} = \frac{1}{\kappa} \int_0^1 v v_t dx,
\]

\[
= \int_0^1 v v_{xx} dx
\]

\[
= v v_x|_0^l - \int_0^1 v_x^2 dx
\]

Since \( v(0, t) = v(l, t) = 0, \) we have

\[
\frac{dE}{dt} = -\int_0^1 v_x^2 dx \leq 0,
\]

i.e. \( E \) is a decreasing function of \( t. \) Also, by definition, \( E(t) \) is nonnegative and from the condition \( v(x, 0) = 0 \) we have \( E(0) = 0. \) Therefore

\[
E(t) \equiv 0 \ \forall t > 0 \Rightarrow v(x, t) \equiv 0 \ \text{in} \ 0 \leq x \leq l, \ t \geq 0.
\]

Hence the theorem.