Math 273: Spring 2013

Exam II Practice

How to study for your Math 273 Exam:
- Reread all lecture notes and go over lecture examples.
- Go over all homework assigned and use the HW keys to check your work.
- Use the practice problems below and the ones from your text for more practice. Answers to odd problems are in the back of your text book.

Exam questions will be similar to those covered in lecture and homework and will require the same knowledge and understanding used for homework problems. If you have done all your homework and reviewed the lecture notes (with understanding), there should be no surprises on the exam. Remember, no calculators and no notes on the exam.

The following are the formulas and blank unit circle exactly as you will see it on the exam. Nothing else will be given, if you do not see something you need here, it is assumed that you have it MEMORIZED.

FORUMLAS

Polar/Cylindrical Coordinates

\[ x = r \cos(\theta) \]
\[ y = r \sin(\theta) \]
\[ r^2 = x^2 + y^2 \]
\[ \tan(\theta) = \frac{y}{x} \]
\[ dA = r \, dr \, d\theta \]
\[ dV = r \, dr \, d\theta \, dz \]

Spherical Coordinates

\[ r = \rho \sin(\phi) \]
\[ x = \rho \sin(\phi) \cos(\theta) \]
\[ y = \rho \sin(\phi) \sin(\theta) \]
\[ z = \rho \cos(\phi) \]
\[ \rho^2 = x^2 + y^2 + z^2 \]
\[ dV = \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi \]

Jacobian

\[ \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} \]

Graphs of Quadric Surfaces

Ellipsoid:
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

Elliptic Paraboloid:
\[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

Hyperbolic Paraboloid:
\[ \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \]

Hyperbolic Paraboloid of One Sheet:
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

Cone:
\[ \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

Hyperboloid of Two Sheets:
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]
Section 11.7: Maximum and Minimum Values
- Use the Second Derivative Test to find local extrema and saddle points.

**Ex:** Let \( f(x, y) = x^3y - 12x^2 - 8y \).

  a) Find all critical points of \( f \).
  b) Calculate \( D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \)
  c) Use the Second Derivative Test to identify the local maximum and minimum value(s) (if any) and saddle point(s) (if any) of the function.

Section 12.1: Double Integrals Over Rectangles
- Set up and evaluate iterated integrals
  - Understand and apply Fubini’s Theorem

**Ex:** Calculate the iterated integral: \( \int_{0}^{1} \int_{0}^{1} xy\sqrt{x^2 + y^2} \, dy \, dx \).

Section 12.2: Double Integrals Over General Regions
- Identify, set up, and evaluate Type I and Type II region iterated integrals
  - Switch the order of integration

**Ex:** Find the volume of the solid which is above the region in the \( xy \)-plane bounded by \( x = 0 \), \( y = 0 \), \( x = \sqrt{y} \), \( y = 8 \), and below the function \( f(x, y) = e^{-x^2} \). *(Hint: Choose the order of integration carefully!)*

Section 12.3: Double Integrals in Polar Coordinates
- Switch an integral from Cartesian coordinates to Polar coordinates and evaluate.

**Ex:** Use polar coordinates to find the volume of the solid above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 1 \).

Section 12.5: Triple Integrals
- Set up and evaluate triple integrals
  - Switch the order of integration

**Ex:** Evaluate \( \iiint_{E} 6xy \, dV \), where \( E \) lies under the plane \( z = 1 + x + y \) and above the region in the \( xy \)-plane bounded by the curves \( y = \sqrt{x} \), \( y = 0 \), and \( x = 1 \).

Section 12.6: Triple Integrals in Cylindrical Coordinates
- Switch an integral from Cartesian coordinates to Cylindrical coordinates and evaluate.

**Ex:** Use cylindrical coordinates to evaluate \( \iiint_{E} x^2 \, dV \) where \( E \) is the solid that lies within the cylinder \( x^2 + y^2 = 1 \), above the plane \( z = 0 \), and below the cone \( z^2 = 4x^2 + 4y^2 \).
Section 12.7: Triple Integrals in Spherical Coordinates
- Switch an integral from Cartesian coordinates to Spherical coordinates and evaluate.

**Ex:** Use spherical coordinates to find the volume of the solid that lies within the sphere \( x^2 + y^2 + z^2 = 25 \), above the \( xy \)-plane, and below the cone \( z = \sqrt{x^2 + y^2} \).

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Section 12.8: Change of Variables in Multiple Integrals
- Switch an integral in one coordinate system to another given a transformation

**Ex:** Use the transformation \( x = 2u \), \( y = 3v \) to evaluate the integral \( \iiint_R x^2 \, dA \), where \( R \) is the region bounded by the ellipse \( 9x^2 + 4y^2 = 36 \).

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Section 13.1: Vector Fields
- Sketch, find, or match vector (and gradient) fields

**Ex:** Find the gradient vector field of \( f(x,y) = \ln(x+2y) \).

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Section 13.2: Line Integrals
- Set up and evaluate line integrals along a smooth curve, in space, or of vector fields

**Ex:** Evaluate the line integral \( \int_C x^2 \, dx + y^2 \, dy + z^2 \, dz \) where \( C \) is the curve \( x=t, y=t^2, z=t^3, 0 \leq t \leq 1 \).
Let $f(x, y) = x^3 y - 12x^2 - 8y$.

a) Find all critical points of $f$.

\[
f_x(x, y) = 3x^2 y - 24x = 0 \quad f_y(x, y) = x^3 - 8 = 0
\]

\[
3x (xy - 8) = 0 \\
x \cdot y = 8 \\
2y = 8 \\
y = 4
\]

Critical Point: $(2, 4)$

b) Calculate $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

\[
f_{xx}(x, y) = 6xy - 24 \\
f_{yy}(x, y) = 0 \\
f_{xy}(x, y) = 3x^2
\]

\[
D = (6xy - 24)(0) - (3x^2)^2 = -9x^4
\]

c) Use the Second Derivative Test to identify the local maximum and minimum value(s) (if any) and saddle point(s) (if any) of the function.

\[
D(2, 4) = -9(2)^4 < 0
\]

\[
f(2, 4) = -48 \text{ is a saddle point}
\]
Ex: Calculate the iterated integral: \( \int_0^1 \int_0^1 xy \sqrt{x^2 + y^2} \, dy \, dx \).

\[
\int_0^1 \int_0^1 xy \sqrt{x^2 + y^2} \, dy \, dx = \int_0^1 \left. \frac{x}{2} \left( \frac{2}{3} \sqrt{x^2 + y^2} \right)^{3/2} \right|_0^1 \, dx \\
= \int_0^1 \left[ \frac{x}{3} \left( \frac{x^2 + 1}{3} \right)^{3/2} - \frac{x^4}{3} \right] \, dx \\
= \left. \left[ \frac{1}{3} \left( \frac{2}{5} \frac{x^2 + 1}{5} - \frac{x^5}{5} \right) \right] \right|_0^1 \\
= \left. \left[ \frac{1}{15} \sqrt{2} - \frac{1}{15} \right] \right|_0^1 \\
= \frac{2}{15}\left( \sqrt{2} - 1 \right)
\]

Ex: Find the volume of the solid which is above the region in the xy-plane bounded by \( x = 0 \), \( y = 0 \), \( x = \frac{y}{\sqrt{3}} \), \( y = \frac{2}{\sqrt{3}} \) and below the function \( f(x, y) = e^{x^4} \). (Hint: Choose the order of integration carefully!)

\[
\int_0^2 \int_0^{\sqrt[3]{x^3}} e^{x^4} \, dy \, dx \\
= \int_0^2 ye^{x^4} \bigg|_{y=0}^{y=x^3} \, dx \\
= \int_0^2 x^3 e^{x^4} \, dx \\
= \left. \frac{1}{4} e^{x^4} \right|_0^2 \\
= \frac{1}{4} \left( e^{16} - 1 \right)
\]
Ex: Use polar coordinates to find the volume of the solid above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 1 \).

Intersection: \( x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 1 \)
\( x^2 + y^2 = \frac{1}{2} \)

\[ V = \iiint \left( \sqrt{1-x^2-y^2} - \sqrt{x^2+y^2} \right) dA \]
\[ x^2+y^2 \leq \frac{1}{2} \]

\[ = \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{2}}} (\sqrt{1-r^2} - r) r \, dr \, d\theta \]

\[ = 2\pi \int_0^{\sqrt{\frac{1}{2}}} [r\sqrt{1-r^2} - r^2] \, dr \]

\[ = \frac{2\pi}{3} \left[ -\frac{1}{2} \right] \left. \frac{3}{2} (1-r^2)^{3/2} - \frac{r^3}{3} \right|_{r=0}^{1/\sqrt{2}} \]

\[ = \frac{2\pi}{3} \left( -\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{12} \right) - \left( -\frac{1}{3} \right) = \frac{4\pi}{3} \left( 2-\sqrt{2} \right) \]

Ex: Evaluate \( \iiint_B 6xy \, dV \), where \( E \) lies under the plane \( z = 1+x+y \) and above the region in the \( xy \)-plane bounded by the curves \( y = \sqrt{x} \), \( y = 0 \), and \( x = 1 \).

\[ \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz \, dy \, dx \]

\[ = \int_0^1 \int_0^{\sqrt{x}} (6xy) \left. \right|_{z=0}^{z=1+x+y} \, dy \, dx \]

\[ = \int_0^1 \int_0^{\sqrt{x}} (6xy(1+x+y)) \, dy \, dx \]

\[ = \int_0^1 \int_0^{\sqrt{x}} (6xy + 6x^2y + 6xy^2) \, dy \, dx \]

\[ = \int_0^1 3xy^2 + 3x^2y^2 + 2xy^3 \left. \right|_{y=0}^{y=\sqrt{x}} \, dx \]

\[ = \int_0^1 (3x^2 + 3x^3 + 2x^{9/2}) \, dx \]

\[ = \left. \frac{3x^3}{3} + \frac{3x^4}{4} + \frac{4}{7} x^{7/2} \right|_0^1 = 1 + \frac{3}{4} + \frac{4}{7} = \frac{65}{28} \]
Ex: Use cylindrical coordinates to evaluate \( \iiint_E x^2 \, dV \) where \( E \) is the solid that lies within the cylinder \( x^2 + y^2 = 1 \), above the plane \( z = 0 \), and below the cone \( z^2 = 4x^2 + 4y^2 \). \( \left( \frac{z^2}{2^2} = \frac{x^2 + y^2}{4} \right) \)

\[
\iiint_E x^2 \, dV = \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2(\theta) \cdot r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 r^3 \cos^2(\theta) \, dz \, dr \, d\theta - \int_0^{2\pi} \int_0^1 2r^4 \cos^2(\theta) \, dr \, d\theta
\]

\[
= \frac{2}{5} \int_0^{2\pi} \int_0^1 r^5 \cos^2(\theta) \, dr \, d\theta = \int_0^{2\pi} \frac{2}{5} \cos^2(\theta) \, d\theta
\]

\[
= \frac{1}{5} \left[ 2\pi + \frac{1}{2} \sin(2\theta) \right]_0^{2\pi} = \frac{2\pi}{5}
\]

Ex: Use spherical coordinates to find the volume of the solid that lies within the sphere \( x^2 + y^2 + z^2 = 25 \), above the xy-plane, and below the cone \( z = \sqrt{x^2 + y^2} \).

\[
V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^5 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta
\]

\[
= 2\pi \int_{\pi/4}^{\pi/2} \left[ \frac{\rho^3}{3} \sin(\phi) \right]_0^5 \, d\phi
\]

\[
= 2\pi \cdot \frac{125}{3} \int_{\pi/4}^{\pi/2} \sin(\phi) \, d\phi
\]

\[
= \frac{250\pi}{3} \left[ -\cos(\phi) \right]_{\pi/4}^{\pi/2} = \frac{250\pi}{3} \frac{\sqrt{2}}{2} = \frac{250\sqrt{2}}{6}
\]
Use the transformation \( x = 2u, \ y = 3v \) to evaluate the integral \( \iiint_R x^2 \, dA \), where \( R \) is the region bounded by the ellipse \( 9x^2 + 4y^2 = 36 \).

\[
9x^2 + 4y^2 = 9(2u)^2 + 4(3v)^2 = 36
\]

\[
9(4)u^2 + 4(9)v^2 = 36
\]

\[
\Rightarrow u^2 + v^2 = 1
\]

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6
\]

\[
\iiint_R x^2 dA = \iiint_R (2u)^2 (6) du \, dv
\]

\[
\text{with } u^2 + v^2 \leq 1
\]

\[
= 24 \int_0^{2\pi} \int_0^1 r^3 \cos^2(\theta) \, dr \, d\theta
\]

\[
= 24 \int_0^{2\pi} \left[ \frac{r^4}{4} \cos^2(\theta) \right]_{r=0}^1 \, d\theta
\]

\[
= 6 \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta
\]

\[
= 3 \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_0^{2\pi} = 6\pi
\]
Ex: Find the gradient vector field of \( f(x, y) = \ln(x + 2y) \).

\[
\nabla f(x, y) = \frac{1}{x + 2y} \hat{i} + \frac{2}{x + 2y} \hat{j}
\]

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Ex: Evaluate the line integral \( \int_C x^2 \, dx + y^2 \, dy + z^2 \, dz \) where \( C \) is the curve \( x = t, y = t^2, z = t^3, 0 \leq t \leq 1 \).

\[
\begin{align*}
x &= t, & \ dx &= dt \\
y &= t^2, & \ dy &= 2t \ dt \\
z &= t^3, & \ dz &= 3t^2 \ dt
\end{align*}
\]

\[
\int_C x^2 \, dx + y^2 \, dy + z^2 \, dz = \int_0^1 \left[ t^2 + t^2(2t) + t^3(3t^2) \right] dt
\]

\[
= \int_0^1 (t^2 + 2t^3 + 3t^5) \, dt
\]

\[
= \left. \frac{t^3}{3} + \frac{t^4}{2} + \frac{t^6}{2} \right|_0^1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = \frac{4}{3}
\]