Order Topology

Topologies often arise naturally from other types of structures, such as in analysis where almost all topologies are derived from metrics, norms and inner products. Order structures also induce topologies in various ways, but here we consider only one: the order topology on a simply ordered set. For any simply ordered set \((X, \leq)\), the interval notations used in \((\mathbb{R}, \tau_u)\) extend to \(X\) in a fairly obvious way:

\[
[a, b] = \{ x \in X \mid a \leq x \leq b \};
\]
\[
(a, b) = \{ x \in X \mid a < x < b \};
\]
\[
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\]
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(a, b] = \{ x \in X \mid a < x \leq b \};
\]

A basis \(B\) for the order topology \(\tau_o\) on \(X\) is defined by considering 4 cases:

(a) If \(X\) has no least element and no greatest element, then

\[
B = \{(a, b) \mid a, b \in X, \ a < b\};
\]

(b) If \(X\) has a least element \(a_0\) but no greatest element, then

\[
B = \{(a, b) \mid a, b \in X, \ a < b\} \cup \{[a_0, b) \mid b \in X\};
\]

(c) If \(X\) has a greatest element \(b_0\) but no least element, then

\[
B = \{(a, b) \mid a, b \in X, \ a < b\} \cup \{(a, b_0] \mid a \in X\};
\]

(d) If \(X\) has a least element \(a_0\) and a greatest element \(b_0\), then

\[
B = \{(a, b) \mid a, b \in X, \ a < b\} \cup \{[a_0, b) \mid b \in X\} \cup \{(a, b_0] \mid a \in X\}.
\]

In every case, it is clear that \(B\) covers \(X\), and any non-empty intersection of two sets in \(B\) is again a set in \(B\), so \(B\) is a basis for a topology on \(X\).

Examples

1. \(X = \mathbb{R}\), with the usual simple ordering. Here, we are in case (a), and \(\tau_o = \tau_u\).
2. $X = [0, 1]$, with the usual simple ordering. Here, we are in case (d). Note that for $0 < x < 1$, $\mathcal{B}_x = \{(x - \epsilon, x + \epsilon) \mid 0 < \epsilon < \min\{x, 1-x\}\}$ is a basic $\tau_o$-nbhd system at $x$. $\mathcal{B}_0 = \{(0, \epsilon) \mid 0 < \epsilon < 1\}$ and $\mathcal{B}_1 = \{(\epsilon, 1) \mid 0 < \epsilon < 1\}$ are basic $\tau_o$-nbhd systems at 0 and 1, respectively. Note that $[0, 1)$ is open in the order topology on $[0, 1]$, but not in the order topology on $\mathcal{R}$.

3. $X = \mathbb{R}^2$ with the dictionary order. In this case, we shall contrast the order topology $\tau_o$ with the usual topology on $\mathbb{R}^2$ used in analysis. The latter, call it $\tau_u$, has a base of “open disks” of the form $B((a,b), \epsilon) = \{(x,y) \mid (x-a)^2 + (y-b)^2 < \epsilon^2\}$, for all $(a,b) \in \mathbb{R}^2$ and $\epsilon > 0$.

Recall the dictionary order on $\mathbb{R}^2$ is defined by

$$(a,b) \leq (c,d) \text{ iff } (a < c \text{ or } a = c \text{ and } b \leq d).$$

$\mathbb{R}^2$ has neither a least nor a greatest element under this order, so we are in case (a). Here we encounter some notational problems between the “ordered pair” notation and the “open interval” notation, since in the present case, the elements in the open intervals are ordered pairs. Nonetheless, we can deduce that in $(\mathbb{R}, \text{dictionary order})$, the open interval between any pair $(a,b)$ and any other pair $(c,d)$ will assume one of two forms (which we’ll call “type 1” and “type 2”), depending on whether $a = c$ and $b < d$ or $a < c$:

![Diagram showing types 1 and 2 intervals](image-url)
Note that every type 2 open interval is a union of type 1 open intervals. Thus, the collection of all type 1 open intervals in $\mathbb{R}^2$ is a $\tau_o$-basis. This basis can be described by defining the type 1 open interval of points between the points $(a, b)$ and $(a, c)$ as $V(a, b, c) = \{(a, x) \mid b < x < c\}$, and then describing the basis $\mathcal{B}'$ as

$$\mathcal{B}' = \{V(a, b, c) \mid a, b, c \in \mathbb{R} \text{ and } b < c\}$$

To compare the usual topology on $\mathbb{R}^2$ with $\tau_o$, observe that every open disk $B((a, b), \epsilon)$ is a union of sets in $\mathcal{B}'$. It follows that $\tau_u \subseteq \tau_o$. Furthermore, no member of $\mathcal{B}'$ can be written as a union of open disks, so we have $\tau_u < \tau_o$.

4. $X = S_\Omega$. Recall that $S_\Omega$ is an uncountable well-ordered set such that each of its initial sections in countable. Our discussion of ordinal numbers led to an equivalent interpretation of $S_\Omega$ as the set of all countable ordinals in their natural ordering, relative to which 0 is the least element of $S_\Omega$, and there is no greatest element. Other relevant facts about $S_\Omega$ include:

(1) Each ordinal $\lambda \in S_\Omega$ has an immediate successor $\lambda + 1$;

(2) Any countable collection of countable ordinals has a least upper bound which is also a countable ordinal.

Observe that $[0, \lambda)$ is $\tau_o$-open for all $\lambda > 0$, and since this holds when $\lambda = 1$, the set $\{0\}$ is $\tau_o$-open. If $\lambda$ is any non-limit ordinal, then $\{\lambda\} = (\lambda - 1, \lambda + 1)$ is a $\tau_o$-open set. For a limit ordinal $\lambda$, a basic nbhd system at $\lambda$ consists of sets of the form $(\mu, \lambda + 1) = [\mu + 1, \lambda]$, where $\mu < \lambda$. For all non-limit ordinals, including 0, the basic nbhd system at $\lambda$ is the singleton set $\{\lambda\}$.

Note that there are uncountably many limit ordinals in $S_\Omega$. To see this, suppose not. Then the countable set of limit ordinals would have a least upper bound, $\mu$, which must also be countable. But then $\mu + n$ is countable for all $n \in \mathbb{N}$, and the countable collection $\{\mu + n \mid n \in \mathbb{N}\}$ would have a least upper bound, which would have to be a limit ordinal greater than $\mu$. 
**Isolated points**

In general, a point \( x \) in a topological space \((X, \tau)\) is called an **isolated point** iff the set \( \{x\} \) is \( \tau \)-open. As we’ve just seen, the isolated points in \((S_\Omega, \tau_o)\) are precisely the non-limit ordinals.

A topological space is called **discrete** iff every point in the space is isolated.

**Closed Sets**

A subset \( A \) in a topological space \((X, \tau)\) is defined to be **closed** (or \( \tau \)-closed) iff \( X - A \in \tau \). Let \( \gamma_\tau = \{A \subseteq X \mid X - A \in \tau\} \) be the collection of all \( \tau \)-closed sets. Using the DeMorgan laws, it is relatively easy to see that \( \gamma_\tau \) satisfies the following three axioms, which are "dual" to the open-set axioms:

1. \( \emptyset, X \in \gamma_\tau; \)
2. If \( \{A_i \mid i \in I\} \subseteq \gamma_\tau \), then \( \bigcap_{i \in I} A_i \in \gamma_\tau; \)
3. If \( A, B \in \gamma_\tau \), then \( A \cup B \in \gamma_\tau. \)

More generally, any collection of subsets of \( X \) satisfying these three "closed set axioms" will define the collection of closed sets for some topology on \( X \); namely, the topology consisting of the complements of the closed sets.

**Prop. N6.1**  Let \((X, \tau_o)\) be any simply ordered set with its order topology. If \( a \leq b \) in \( X \), then \([a, b]\) is a closed set.

**Proof:** We must show \( X - [a, b] \) is \( \tau_o \)-open. Note that if we define \( \mathcal{U} = \{x \in X \mid x < a\} \) and \( \mathcal{V} = \{x \in X \mid x > b\} \), then we can write \( X - [a, b] = \mathcal{U} \cup \mathcal{V} \). If we show that \( \mathcal{U} \) and \( \mathcal{V} \) are open, then it follows that \( X - [a, b] \), being the union of two open sets, is also open.

Now, if \( X \) has a least element \( a_0 \), then \( \mathcal{U} = [a_0, a) \) is open by definition of \( \tau_o \). If \( X \) has no least element, then \( \mathcal{U} = \{x \in X \mid x < a\} = \bigcup_{y < a} (y, a) \), which is open because it is a union of \( \tau_o \)-open sets. Similar reasoning shows that \( \mathcal{V} \) is open, and hence \( X - [a, b] = \mathcal{U} \cup \mathcal{V} \) is open. ■