13.4 (6) \[ F = \left< \frac{p}{x}, \frac{q}{y} \right> \]

C: \[ \text{clockwise} \]

\[ \gamma \]

\[ (x-2)^2 + (y-3)^2 = 1 \]

\[ \oint_C F \cdot dr = \iint_D \left( \frac{q_x - p_y}{x^2 + y^2} \right) \, dA \]

\[ \begin{array}{c}
\int_S \int_D 2 \cdot \frac{1}{1 + \left( \frac{y}{x} \right)^2} \left[ -\frac{2y}{x^2 + y^2} \right] - \left( 1 - \frac{2y}{x^2 + y^2} \right) \, dA \\
= \int_S \oint \, dA = \int_D \int_D (-1) \, dA \\
= - (\text{area of } D) = -\pi (1)^2 = -\pi.
\end{array} \]
13.5 (14) \( F = \langle \text{e}^z, 1, xz^2 \rangle \)
\[
? \quad \Rightarrow \quad \langle f_x, f_y, f_z \rangle
\]
for some \( f \)?

\[
f = xe^z + y. \quad (\text{is conservative})
\]

\[
\int_C f_x \, dx + f_y \, dy
\]

17.3 (18) \( F = \langle 1-ye^{-x}, e^{-x} \rangle \)
\[
= \nabla (x + ye^{-x}),
\]
on all of \((\mathbb{R}^2)\). So this integral is path independent on \((\mathbb{R}^2)\), and:
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2) - f(0, 1) = \frac{1}{(1 + 2)} = \frac{1}{3}
\]
\[ \mathbf{F} = \left< \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right> \]

a) \[ P_y = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} \]
\[ = \frac{y^2-x^2}{(x^2+y^2)^2} \]

b) \[ Q_x = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} \]
\[ = \frac{y^2-x^2}{(x^2+y^2)^2} \]

\[ \begin{align*}
&x = \cos t, 0 \leq t \leq \pi \\
y = \sin t
\end{align*} \]
\[ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \left\langle \frac{-\sin t}{1}, \frac{\cos t}{1} \right\rangle \cdot \left\langle -\sin t, \cos t \right\rangle \, dt \]

\[ = \int_0^\pi \frac{-\sin t \cdot (-\sin t)}{1} + \frac{\cos t \cdot \cos t}{1} \, dt \]

\[ = \int_0^\pi \frac{\sin^2 t + \cos^2 t}{1} \, dt \]

\[ = \int_0^\pi 1 \, dt = \pi. \]

probably \[ \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\pi. \]

\[ \text{does not contradict Thm 6,} \]

because there is not a simple region \( D \)
which contains the curve where \( Q_x, Q_y \) exist and are not continuous,
\[ Q_x = 0 \text{ at } (0,0) \]
which is inside the curve.
Suppose $\vec{F}$ represents a fluid's velocity, in cm/sec. So $\vec{F}(x_0, y_0, z_0) = \langle 1, 2, 3 \rangle$ means that at $(x_0, y_0, z_0)$, fluid is flowing in the direction of $\langle 1, 2, 3 \rangle$ at a speed of $|\langle 1, 2, 3 \rangle| = \sqrt{14}$ cm/sec. Now, put a little cube around $(x_0, y_0, z_0)$ with two faces normal to $\vec{F}$, and side length $a$.

Call the surface of this cube $S$, oriented outward.
What does it mean if \( \int_{S} \mathbf{F} \cdot d\mathbf{s} \) is negative? It means that overall, fluid is entering the cube with more velocity than it is leaving with. Since \((\text{mass in}) = (\text{mass out})\), it follows that the fluid leaving the cube is more dense than that entering. In other words, the fluid is being \underline{compressed} within the cube. To measure whether a fluid (or in general a field) is being compressed at a single point,
can define
\[
\begin{bmatrix}
\text{divergence of } \mathbf{F} \\
\text{at } (x_0, y_0, z_0)
\end{bmatrix} = \text{div} \mathbf{F}(x_0, y_0, z_0)
\]
\[
= \lim_{a \to 0} \left( \frac{\iiint_S \mathbf{F} \cdot \mathbf{n} \, dV}{a^3} \right), \text{ where } S
\]
is the previously described cube.

This is the limit of the net outward flux per unit volume, as the size of the cube approaches 0. It can be shown that this limit is equal to
\[
\frac{\partial P}{\partial x}(x_0, y_0, z_0) + \frac{\partial Q}{\partial y}(x_0, y_0, z_0) + \frac{\partial R}{\partial z}(x_0, y_0, z_0).
\]
So we have \((F = \langle p, q, r \rangle)\)

\[
\text{div } F = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}
= \nabla \cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \langle p, q, r \rangle
\]

At \((x_0, y_0, z_0)\):

\[
\text{div } F > 0 \text{ means a small cube at that point would have more net outward flux than inward;}
\]

\[
\text{div } F < 0 \text{ means more net inward flux;}
\]

\[
\text{div } F = 0 \text{ means equal tendency of } F \text{ to flow in and out of cube at } (x_0, y_0, z_0)
\]

\[
\text{div } F = 0 \text{ throughout a region } D \text{ means } F \text{ is "incompressible" in } D.
\]
Last 2 Theorems:

Stokes' Thm (p. 786):

In this situation in $\mathbb{R}^3$:

- "bubbled" surface $S$, oriented up
- boundary curve $C$, oriented ccw from above

$F = \langle P, Q, R \rangle$

$$\oint_C F \cdot dr = \iint_S \text{curl}(F) \cdot dS$$
Divergence Theorem

In a situation in $\mathbb{R}^3$ like this:

- closed surface $\Sigma$
- oriented outward

$\Sigma$ encloses solid region $E$

\[ \int_{\Sigma} \mathbf{F} \cdot d\mathbf{s} = \iiint_E \text{div}(\mathbf{F}) \, dV. \]
13.9 (5) \[ \vec{F} = \langle e^x \sin y, e^x \cos y, yz^2 \rangle \]

E:

\[ \text{box surface is } S, \quad \text{solid is } E. \]

Find \[ \iiint_S \vec{F} \cdot d\vec{S}. \]

Div. Thm

\[ \iiint_E \text{div}(\vec{F}) \, dV \]

\[ = \iiint_S \left( \rho_x + \rho_y + \rho_z \right) \, dA \]

\[ = \iiint_E \left( e^x \sin y + e^x \cos y + 2yz^2 \right) \, dxdydz \]

\[ = \iiint_E 2yz^2 \, dxdydz \]