11.3 (39) \( x - z = \arctan(yz) \)

To find \( \frac{\partial z}{\partial x} \), treat \( z \) as function of \( x \), \( y \) as a constant. Differentiate:

\[
1 - \frac{\partial^2 z}{\partial x^2} = \frac{1}{1 + (yz)^2} \cdot \frac{\partial}{\partial x} (yz) \\
1 - \frac{\partial^2 z}{\partial x^2} = \frac{1}{1 + y^2 z^2} \cdot \frac{y \frac{\partial z}{\partial x}}{\partial x} \\
1 = \left( \frac{y}{1 + y^2 z^2} + 1 \right) \frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial x} = \frac{1}{y} \frac{\partial}{\partial x} \left( \frac{1}{1 + y^2 z^2} + 1 \right) \\
= \frac{1 + y^2 z^2}{1 + y^2 z^2 + y}
\]
11.4 (24) \[ z = x^2 - xy + 3y^2, \quad f(x,y) \]

(\(x, y\)) \( (3, -1) \) to \( (2.96, -0.95) \)
\[ \Delta x = -0.04 \quad \Delta y = 0.05 \]

\[ \Delta z = f(\text{final in.}) - f(\text{initial}) \]
\[ = f(2.96, -0.95) - f(3, -1) \]
\[ = ? - 15 = \square \]

\[ dz = f_x(3, -1) \cdot \Delta x + f_y(3, -1) \cdot \Delta y \]
\[ f_x = 2x - y, \quad f_y = -x + 6y \]
\[ f_x(3, -1) = 7 \quad f_y(3, -1) = -9 \]

\[ dz = 7(-0.04) + -9(0.05) \]
\[ = -0.28 + -0.45 = \square \]

\[ = -0.73 \]
11.6 Directional Derivative

slope in t-direction is what we're after

slope $f_x(a,b)$

$z = f(x,y)$

trace above t-axis

slope $f_y(a,b)$

$(a,b)$

$u = \langle u_1, u_2 \rangle$

(unit vector)

$u_1^2 + u_2^2 = 1$

line is:

\[\begin{align*}
x &= a + u_1 t \\
y &= b + u_2 t
\end{align*}\]
The trace that sits above the "t" axis:

\[ z = f(x_1, y_1) = f(a + u_1 t, b + u_2 t) \]

slopes here is

\[
\left. \frac{dz}{dt} \right|_{t=0} = f_x(a, b) \cdot \frac{dx}{dt} + f_y(a, b) \cdot \frac{dy}{dt} = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2
\]

= \langle f_x(a, b), f_y(a, b) \rangle \circ \langle u_1, u_2 \rangle.

"gradient of \( f \) at \((a, b)\)."
**Some defs:**

- \( \nabla \) is "del" operator.
  - In 2-D: \( \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \)
  - In 3-D: \( \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \).

\( \nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \text{gradient of } f \) at \((x,y)\)

---

**Directional Derivative of** \( f \) **at** \((a,b)\),

**in direction of unit vector** \( \bar{u} \) **is:**

\[
D_{\bar{u}} f(a,b) = \nabla f(a,b) \cdot \bar{u}
\]

\[
= \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle
\]

\[
= f_x(a,b)u_1 + f_y(a,b)u_2.
\]

**Similarly, for** \( f(x,y,z) \),

**\( D_{\bar{u}} f(a,b,c) = \text{direct. deriv. of } f \) at \((a,b,c)\),**

**in direction of** \( \bar{u} \).
\[ = \nabla f(a, b, c) \cdot \vec{u} \]
\[ = \left( f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \right) \cdot \left( u_1, u_2, u_3 \right) \]

(\( \vec{u} \) must be a unit vector to give proper answer)

Q: given \( f(x, y, z) \), input \( (a, b, c) \), what direction \( \vec{u} \) should I change my input in to most rapidly increase the value of \( f \)?

A: \[ \frac{\partial f}{\partial u}(a, b, c) = \left( f_x, f_y, f_z \right) \cdot \vec{u} \]
\[ = \frac{\left| \left( f_x, f_y, f_z \right) \right|}{\left| \vec{u} \right|} \cos(\theta) \]

fixed \( \theta \)

So should choose \( \theta = 0 \) i.e. choose \( \vec{u} \) in the same direction as \( \nabla f(a, b, c) \).
Note that when I chose \( \mathbf{u} \) in direct. of \( \nabla f(a, b, c) \), I get
\[
D_{\mathbf{u}} f(a, b, c) = \left| \nabla f(a, b, c) \right|.
\]

11. 6. 4 \( f(x, y) = y \ln(x) \), \( P(1, -3) \), \( \mathbf{u} = \left< \frac{2}{3}, \frac{3}{3} \right> \).
\[\text{a) } \nabla f(x, y) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> = \left< \frac{y}{x}, \ln(x) \right>.\]
\[\text{b) } \nabla f(1, -3) = \left< -\frac{3}{3}, \ln(1) \right> = \left< -1, 0 \right>.\]

\( \text{c) } D_{\mathbf{u}} f(1, -3) = \nabla f(1, -3) \cdot \mathbf{u} \)
\[\left< -1, 0 \right> \cdot \left< -\frac{4}{3}, \frac{3}{3} \right> = \frac{12}{5} = 2.4 \]
11.6 (10) \( f(x, y, z) = \frac{x}{y+z} \), \( \nabla = \langle 1, 2, 3 \rangle \).

\[ \nabla f = \left\langle \frac{1}{y+z}, \frac{-x}{(y+z)^2}, \frac{-x}{(y+z)^2} \right\rangle \]

\[ \nabla f(4, 1, 1) = \left\langle \frac{1}{2}, -1, -1 \right\rangle \]

\[ \nabla \cdot f(4, 1, 1) = \left\langle \frac{1}{2}, -1, -1 \right\rangle \cdot \frac{1}{\sqrt{14}} \]

\[ = \left(1 \left(\frac{1}{2}\right) + 2(-1) + 3(-1)\right) \frac{1}{\sqrt{14}} = \frac{-9}{2\sqrt{14}} \]

For max rate of change, would use

\[ \vec{u} = \frac{\left\langle 1, -2, -2 \right\rangle}{3} \]

Max rate of change possible is

\[ 1 \left\langle \frac{1}{2}, -1, -1 \right\rangle = \sqrt{\frac{9}{4}} = \frac{3}{2} \]