Some HW3 Solutions

11.2 ⑨ Option 1: Use Squeeze Theorem:

As \((x, y) \to (0, 0)\), \(0 \leq |f(x, y)| = \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \frac{|x| |y|}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{x^2 y^2}}{\sqrt{x^2 + y^2}} = \sqrt{x^2 y^2} \to 0\).

So by Squeeze Theorem, \(\lim_{(x, y) \to (0, 0)} f(x, y) = 0\).

Option 2: \(\varepsilon, \delta\) proof: Let \(\varepsilon > 0\). Want to choose \(\delta\) so that

\[|f(x, y) - 0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.\]

But, as shown above, \(|f(x, y)| \leq \sqrt{x^2 + y^2}\) for all \((x, y) \neq (0, 0)\). So choosing \(\delta = \varepsilon\) does the job, because then

\[0 < \sqrt{x^2 + y^2} < \delta \implies \sqrt{x^2 + y^2} < \varepsilon \implies |f(x, y)| < \varepsilon. \quad \text{So} \quad \lim_{(x, y) \to (0, 0)} f(x, y) = 0.\]

11.2 ⑩ Along path \(x = 0\): \(\lim_{(x, y) \to (0, 0)} \left( \frac{2x^2 y}{x^4 + y^4} \right) = \lim_{y \to 0} \left( \frac{0}{y^2} \right) = 0\)

Along path \(y = x^2\): \(\lim_{(x, y) \to (0, 0)} \left( \frac{2x^2 y}{x^4 + y^4} \right) = \lim_{x \to 0} \left( \frac{2x^2 \cdot x^2}{x^4 + x^4} \right) = 1\)

Two different target levels, so \(\lim_{(x, y) \to (0, 0)} f(x, y) = \text{DNE} \).

11.3 ⑪ \(f(x, y, z, t) = \frac{xy^2}{t^2 + z^2}\)

To find \(\frac{\partial f}{\partial x}\) treat \(y, z, t\) as constants:

\[\frac{\partial f}{\partial x} = \frac{2}{t^2 + z^2} \left( x \cdot (y^2) \right) = \frac{2xy^2}{t^2 + z^2}\]

To find \(\frac{\partial f}{\partial y}\) treat \(x, z, t\) as constants:

\[\frac{\partial f}{\partial y} = \frac{2}{t^2 + z^2} \left( \frac{x}{t^2 + z^2} y \right) = \frac{x}{t^2 + z^2} \cdot 2y = \frac{2xy}{t^2 + z^2}\]

To find \(\frac{\partial f}{\partial z}\) treat \(x, y, t\) as constants:

\[\frac{\partial f}{\partial z} = \frac{2}{t^2 + z^2} \left( (xy^2) \cdot (t^2 + z^2) \right) = (xy^2) \cdot (t^2 + z^2) \cdot 2\]

To find \(\frac{\partial f}{\partial t}\) treat \(x, y, z\) as constants:

\[\frac{\partial f}{\partial t} = \frac{2}{t^2 + z^2} \left( (xy^2) \cdot (t^2 + z^2)^{-1} \right) = \frac{2xy^2}{(t^2 + z^2)^2}\]