SHOW APPROPRIATE WORK or EXPLANATION on each problem for full credit. Box or circle your final answers. Answers need not be simplified. Calculators/note sheets are NOT allowed. Numbers in the [ ] indicate what each problem is worth. Check the board for useful info.

1. For the curve \( \vec{r}(t) = \left( \frac{1}{2} t^{3/2}, \sin(\frac{\pi}{2} t), \cos(\frac{\pi}{2} t) \right) \), do the following:

   a) [6] Assuming this curve represents a particle's position, find the particle's velocity and acceleration at \( t = 9 \).

      \[
      \text{velocity} = \vec{r}'(t) = \left< \frac{3}{4} t^{1/2}, \frac{\pi}{2} \cos(\frac{\pi}{2} t), -\frac{\pi}{2} \sin(\frac{\pi}{2} t) \right>
      \]

      \[
      \text{accel.} = \vec{r}''(t) = \left< \frac{3}{8} t^{1/2}, -\frac{\pi^2}{4} \sin(\frac{\pi}{2} t), -\frac{\pi^2}{4} \cos(\frac{\pi}{2} t) \right>
      \]

      At \( t = 9 \), vel. = \( \left< \frac{3}{8}(9)^{1/2}, \frac{\pi}{2} \cos(9\frac{\pi}{2}), -\frac{\pi}{2} \sin(9\frac{\pi}{2}) \right> = \left< \frac{9}{4}, 0, -\frac{\pi}{2} \right> \)

      accel. = \( \left< \frac{3}{8}(9)^{1/2}, -\frac{\pi^2}{4} \sin(9\frac{\pi}{2}), -\frac{\pi^2}{4} \cos(9\frac{\pi}{2}) \right> = \left< \frac{1}{8}, \frac{\pi^2}{4}, 0 \right> \)

   b) [8] Find the parametric equations of the line which is tangent to the given curve at the point \( (4,0,1) \).

      \[
      \vec{r}(t) = \left< \frac{1}{2} t^{3/2}, \sin(\frac{\pi}{2} t), \cos(\frac{\pi}{2} t) \right> = \left< 4, 0, 1 \right>, \text{ need } \int t^{3/2} \, dt = 4 \Rightarrow t^{3/2} = 8 \Rightarrow t = 4.
      \]

      \[
      \vec{r}'(t) = \left< \frac{3}{4} t^{1/2}, \frac{\pi}{2} \cos(\frac{\pi}{2} t), -\frac{\pi}{2} \sin(\frac{\pi}{2} t) \right> = \left< \frac{3}{4}, \frac{\pi}{2}, 0 \right>.
      \]

      So tangent line at \( (4,0,1) \) is:

      \[
      \begin{aligned}
      x &= 4 + \frac{3}{4} t \\
y &= 0 + \frac{\pi}{2} t \\
z &= 1 + 0 t
      \end{aligned}
      \]

   c) [6] Sketch (as accurately as you can) the piece of curve from \( t = 0 \) to \( t = 4 \). Label initial and final points with their \((x, y, z)\) coordinates.

      \[
      \vec{r}(0) = \left< 0, 0, 1 \right>, \quad \vec{r}(4) = \left< 4, 0, 1 \right>.
      \]

      \[
      y = \sin(\frac{\pi}{2} t) \\
z = \cos(\frac{\pi}{2} t)
      \]

      \[
      \begin{aligned}
      \text{x grows with t like this:} & \\
      \text{spirals towards us, clockwise.}
      \end{aligned}
      \]

      So:

      \[
      (4,0,1)
      \]
2. For the function \( f(x, y) = \sqrt{y^2 - 2x} \), do the following:

a) [6] Find and graph the domain of \( f \) in the \( xy \) plane.

\[
\begin{align*}
\sqrt{y^2 - 2x} & \geq 0 \implies y^2 \geq 2x \\
\frac{1}{2}y^2 & \geq x
\end{align*}
\]

b) [12] Draw a contour map of \( f \), showing level curves for \( z \)-levels of 0, 1, 2, 3. Label each contour with its \( z \)-level.

\[
\begin{align*}
\sqrt{y^2 - 2x} = 0 & \implies x = \frac{1}{2}y^2 \\
\sqrt{y^2 - 2x} = 1 & \implies x = \frac{1}{2}y^2 - \frac{1}{2} \\
\sqrt{y^2 - 2x} = 2 & \implies x = \frac{1}{2}y^2 - 2 \\
\sqrt{y^2 - 2x} = 3 & \implies x = \frac{1}{2}y^2 - \frac{9}{2}
\end{align*}
\]

3. [12] Given that \( w = \frac{xy}{z} \), \( x = e^{3t-r^2} \), \( y = rt^2 \), \( z = r \ln(t) \), use the chain rule to find an expression for \( \frac{\partial w}{\partial r} \).

You may leave a mixture of \( x \)'s, \( y \)'s, \( z \)'s, \( r \)'s and \( t \)'s in your answer.

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}
\]

\[
= \left( \frac{y}{z} \right) (-2re^{3t-r^2}) + \left( \frac{x}{z} \right) (t^2) + \left( -\frac{x}{z^2} \right) (\ln(t))
\]
4. [12] Let $f(x, y, z) = (x^2 - z^3)/y$ and let $\vec{v} = \langle -2, 5, 1 \rangle$. Find the directional derivative of the function $f$ at the point $(1, 2, 3)$ in the direction of $\vec{v}$. That is, compute $D_\vec{v}f(1, 2, 3)$, where $\vec{u}$ is a unit vector in the same direction as $\vec{v}$.

$$\nabla f = \langle \frac{2x}{y}, \frac{2x^2 - x^3}{y^2}, -\frac{3z^2}{y} \rangle, \quad \nabla f(1, 2, 3) = \langle 1, \frac{13}{2}, -\frac{27}{2} \rangle,$$

$$\vec{u} = \frac{\langle -2, 5, 1 \rangle}{\sqrt{30}}, \quad \text{so} \quad D_\vec{u}f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \vec{u}$$

$$= \langle 1, \frac{13}{2}, -\frac{27}{2} \rangle \cdot \langle -2, 5, 1 \rangle = \frac{-2 + \frac{65}{2} - \frac{27}{2}}{\sqrt{30}} = \frac{17}{\sqrt{30}}.$$

5. [12] Find an equation for the plane which is tangent to the surface $x^2 e^{x-4z} - 4yz = 15$ at the point $(4, \frac{1}{4}, 1)$.

$$\nabla f = \langle x^2 e^{x-4z} + 2x e^{x-4z}, -4z, -4x^2 e^{x-4z} - 4y \rangle$$

$$\nabla f(4, \frac{1}{4}, 1) = \langle 16e^6 + 8e^6, -4, -64e^6 - 1 \rangle = \langle 24, -4, -65 \rangle$$

So the plane is $24(x-4)-4(y-\frac{1}{4})-65(z-1)=0$. 
6. [12] Find all the critical points \( (x, y) \) for the function \( f(x, y) = 3xy^2 - 2x^3 - 3y^2 \). (you should get 3 of them)

*Hint: you should be able to factor the \( f_y(x, y) = 0 \) equation, which should help*

\[
\begin{align*}
\begin{cases}
    f_x &= 7y^2 - 6x^2 = 0 \\
    f_y &= 6xy - 6y = 0
\end{cases} \quad \Rightarrow \quad \begin{cases}
    y^2 - 2x^2 = 0 \\
    6y(x-1) = 0
\end{cases} \quad \Rightarrow \quad y = 0 \text{ or } x = 1.
\end{align*}
\]

\[
\begin{align*}
    0^2 - 2x^2 &= 0 \\
    y^2 - 2 &= 0
\end{align*}
\]

\[
\begin{align*}
    x &= 0 \\
    y &= \pm\sqrt{2}.
\end{align*}
\]

So critical pts. are \( (0, 0), (1, \sqrt{2}), (1, -\sqrt{2}) \).

7. [12] The function \( f(x, y) = 3x^2 + 3y^2 - 3x^2y - y^3 + 2 \) has critical points at \( (x, y) = (0, 0), (0, 2), (1, 1) \) and \( (1, -1) \). Classify each point as a local maximum, local minimum, or saddle point, and fill in your answers in the table below.

Note that I'm also asking for the \( z \)-coordinate at each point.

<table>
<thead>
<tr>
<th>( (x, y, z) )</th>
<th>Local Extrema of ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, 0, 2) )</td>
<td>min</td>
</tr>
<tr>
<td>( (0, 2, 6) )</td>
<td>max</td>
</tr>
<tr>
<td>( (1, 1, 4) )</td>
<td>saddle</td>
</tr>
<tr>
<td>( (-1, 1, 4) )</td>
<td>saddle</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
    f_x &= 6x - 6xy \\
    f_{xx} &= 6 - 6y \\
    f_y &= 6y - 3x^2 - 3y^2 \\
    f_{yy} &= 6 - 6y \\
    f_{xy} &= f_{yx} &= -6x
\end{align*}
\]

\[
D = (6 - 6y)(6 - 6y) - (-6x)^2
\]

\[
\begin{align*}
    D(0, 0) &= (6)(6) - (0)^2 = \Theta \quad \text{min} \\
    D(0, 2) &= (-6)(-6) - (0)^2 = \Theta \quad \text{max}
\end{align*}
\]

\[
\begin{align*}
    D(1, 1) &= (0)(0) - (6)^2 = \Theta \\
    D(1, -1) &= (0)(0) - (6)^2 = \Theta
\end{align*}
\]