Last time:

\[ \text{[Integral of } f(x,y,z) \text{ over } C] = \int_{C} f(x,y,z) \, ds = \int_{t_0}^{t_1} f(\vec{r}(t), y(t), z(t)) \| \vec{r}'(t) \| \, dt \]

\[ = \int_{t_0}^{t_1} f(\vec{r}(t)) \| \vec{r}'(t) \| \, dt \]

\[ \text{[Integral of } \vec{F}(x,y,z) = \langle P, Q, R \rangle \text{ over } C] = \int_{C} \vec{F} \cdot \frac{\vec{r}'}{\| \vec{r}' \|} \, ds = \int_{C} \vec{F} \cdot d\vec{r} \]

\[ = \int_{C} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]

\[ = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]

\[ ds = \| \vec{r}'(t) \| \, dt \]

\[ d\vec{r} = \vec{r}'(t) \, dt \]
Interpretations:

\[ \int_C f(x,y,z) \, ds = \left[ \text{avg. value of } f \text{ on } C \right] \left[ \text{length of } C \right] \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \left[ \text{avg. value of } \mathbf{F} \cdot \mathbf{T} \text{ on } C \right] \left[ \text{length of } C \right] \]

\[ = \left[ \text{average strength of } \mathbf{F} \text{ in the direction of } \mathbf{T} \text{ on } C \right] \left[ \text{length of } C \right]. \]
Examples

14.2 \( \mathbf{27} \)  Find \( \int_C xyz \, ds \), where \( C \) is:

\[ \int_0^1 (t)(2t)(3t) \sqrt{14} \, dt \]

\[ = 6 \sqrt{14} \int_0^1 t^3 \, dt \]

\[ = \frac{3 \sqrt{14}}{2} \]

or \( \mathbf{\vec{r}}(t) = \langle t, 2t, 3t \rangle \).

\[ ds = |\mathbf{\vec{r}}'(t)| \, dt = (\sqrt{14}) \, dt \]

\[ = \sqrt{14} \, dt. \]
\[ \mathbf{F} = \frac{\langle x(y, z) \rangle}{(x^2 + y^2 + z^2)^{3/2}} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds \]

\[ \mathbf{v}(t) = \langle 1 + 9t, 1 + 9t, 1 + 9t \rangle, \quad 0 \leq t \leq 1, \]

or \[ \mathbf{v}(t) = \langle 1 + 9t, 1 + 9t, 1 + 9t \rangle. \]

\[ \int_C \mathbf{F} \cdot \mathbf{v}'(t) \, dt = \int_C \frac{\langle 1 + 9t, 1 + 9t, 1 + 9t \rangle}{\left( (1 + 9t)^2 + (1 + 9t)^2 + (1 + 9t)^2 \right)^{3/2}} \cdot \langle 9, 9, 9 \rangle \, dt \]

\[ = \int_0^1 \frac{27(1+9t)}{3\sqrt{3}(1+9t)^3} \, dt = ..... = \boxed{} \]
\[ \nabla f = \langle f_x, f_y, f_z \rangle \text{ is a field, so we can integrate it over a curve } \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \ t_0 \leq t \leq t_1: \]

\[ \int_C (\nabla f) \cdot d\vec{r} \]

If we work on this integral a bit, it can be written as

\[ \int_{t_0}^{t_1} \left( \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt} \right) dt \]

which is

\[ \int_{t_0}^{t_1} \left( \frac{df}{dt} \right) dt, \text{ which is } f(x(t_1), y(t_1), z(t_1)) \bigg|_{t_0}^{t_1}, \]

which is \( f(\text{final point}) - f(\text{initial point}) \), or \( f(\vec{r}(t_1)) - f(\vec{r}(t_0)) \).
This means: if curve $C$ lies in some region where $\vec{F}$ is a gradient field with potential function $f$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\nabla f) \cdot d\vec{r} = f(\text{final point}) - f(\text{initial point})$$

$$= \left[ \text{change in potential} \right] = \left[ f(x_0, y_0, z_0) \text{ over } C \right].$$

This is called the Fundamental Theorem (of Calculus) for Line Integrals.

This also means that line integrals of gradient fields depend only on the start- and end-points of the curves, so we say the integrals are path-independent.
If \( \vec{F} = \nabla f \) (for some \( f \)) in this region, then

\[
\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}
\]

\[
= f(x_1, y_1, z_1) - f(x_0, y_0, z_0).
\]

This also means that if \( C \) is a closed curve (end pt. = start pt.) in a region where \( \vec{F} \) is a gradient field, then

\[
\int_{C} \vec{F} \cdot d\vec{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0) = 0. \quad (*)
\]

* when \( C \) is a closed curve, \( \int_{C} \) is often written as \( \oint_{C} \).

The orientation of \( C \) (clockwise or ccw) can be shown too: \( \oint_{C} \).
14.3 (c) \( \vec{F} = \langle z, 1, x \rangle \).

Determine whether \( \vec{F} \) is a gradient field

(BTW, gradient field \( \equiv \) conservative field
\( \equiv \) irrotational field)

Check curl \( \vec{F} \) = \[
\begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z & 1 & x
\end{vmatrix}
= \langle 0, 1, -1 \rangle
\]

= \nabla \times \vec{F}

so this \( \vec{F} \) is gradient/conservative.

To find a potential function, we seek \( f(x, y, z) \)
such that \( \nabla f = \vec{F} = \langle z, 1, x \rangle \). So:

\[
\begin{align*}
 \vec{F}(x, y, z) &= \nabla (xz + y) \\
\frac{\partial}{\partial x} &= z \\
\frac{\partial}{\partial y} &= 1 \\
\frac{\partial}{\partial z} &= x
\end{align*}
\]

So \( \vec{F} = \langle z, 1, x \rangle = \nabla (xz + y) \).

14.3 (43) \( \vec{F} = \langle 2xy + z^2, x^2, 2xz \rangle \),

\( C \) is circle \( \vec{r}(t) = \langle 3 \cos t, 4 \cos t, 5 \sin t \rangle \), \( 0 \leq t \leq 2\pi \).
Find \( \int_C \mathbf{F} \cdot d\mathbf{r} \)

**Note:**

\[
\frac{\partial}{\partial x} \left( x^2 \right) = 2x \\
\frac{\partial}{\partial y} \left( xy + z^2 \right) = y \\
\frac{\partial}{\partial z} \left( z^2 \right) = 2z
\]

\[
\mathbf{F} = (x^2 y + x z^2)
\]

So,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \left. (x^2 y + x z^2) \right|_{(3,4,0)}^{(3,4,0)} = 0.
\]

(as part of HW 13, I've also asked for you to find

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r}, \text{ where } C_1 \text{ is the top half of the circle } (0 \leq t \leq \pi).
\]