If a 2D region $R$ can be described as the region between the lower curve $y = g(x)$ and upper curve $y = f(x)$ from $x = a$ to $x = b$:

Then

$$
\int_R \int h(x,y) \, dA = \int_a^b \int_{g(x)}^{f(x)} h(x,y) \, dy \, dx
$$

**Side note:**

$$
\int_R 1 \, dA = \left( \text{the average value} \right) \left( \text{area of } R \right) = \text{area of } R
$$
If a 2D region $R$ can be described as the region between left curve $x = g(y)$ and right curve $x = f(y)$ from $y = a$ to $y = b$:

Then

$$
\iint_R h(x,y) \, dA
= 
\int_a^b \int_{g(y)}^{f(y)} h(x,y) \, dx \, dy
$$
Some regions do not fit either of these forms, and integration over such regions can require more than one integral:

\[
\iint_R h(x,y) \, dA = \int_a^b \left( \int_{g(x)}^{f_1(x)} h(x,y) \, dy \right) \, dx + \int_{b}^{c} \left( \int_{g(x)}^{f_2(x)} h(x,y) \, dy \right) \, dx
\]
Some regions (like in 13.3) are most conveniently described using polar coordinates \((r, \theta)\):

\[
\begin{align*}
\text{recall:} \\
\begin{array}{l}
\text{y-axis} \\
x^2 + y^2 = r^2 \\
x = r \cos \theta \\
y = r \sin \theta \\
tan \theta = \frac{y}{x}
\end{array}
\end{align*}
\]

\[
R = \{(r, \theta) \mid \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \text{ and } 0 \leq r \leq 3\}
\]
The most general polar regions we'll deal with will fit this form:

\[ R = \{(r, \theta) \mid 0 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{3\pi}{4}\} \]

\[ R \] is the region between inner curve \( r = g(\theta) \) and outer curve \( r = f(\theta) \) from \( \theta = \theta_0 \) to \( \theta = \theta_1 \).

To integrate some function \( h(x, y) \) over such a region, we do this:
\[ \iint_{R} h(x, y) \, dA = \int_{0}^{\theta_{1}} \int_{\theta}^{\phi} h(r \cos(\theta), r \sin(\theta)) \cdot r \, dr \, d\theta \]

\[ \hat{\theta} \text{ needs to represent } dA \]

(a differential area element).

In polar co-ords, \( dA = r \, dr \, d\theta \).

\[ \text{Area} = \int \int dr \, r \, d\theta \]

\[ = r \, dr \, d\theta \]
Find the volume of the solid below \( z = 5 - \sqrt{1 + x^2 + y^2} \) and above \( R = \{(x, y) \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi\} \). 

Volume: 

\[
\int_{\theta=0}^{\pi} \int_{r=0}^{1} (5 - \sqrt{1 + r^2}) \, r \, dr \, d\theta
\]
\[= \int_0^\pi \left( \int_0^1 \left( 5r - r\sqrt{1+r^2} \right) \, dr \right) \, d\theta.\]

Favor: \[\left[ \frac{5}{2} r^2 - \frac{1}{3} (1+r^2)^{\frac{3}{2}} \right]_0^1 = \left( \frac{5}{2} - \frac{1}{3} (2)^{\frac{3}{2}} \right) - \left( 0 - \frac{1}{3} (1)^{\frac{3}{2}} \right)\]

\[= \frac{5}{2} - \frac{2\sqrt{2}}{3} + \frac{1}{3} = \frac{17}{6} - \frac{2\sqrt{2}}{3}\]

Outer: \[\int_0^\pi \frac{17 - 4\sqrt{2}}{6} \, d\theta = \frac{17 - 4\sqrt{2}}{6} \pi\]

can be written as \[\int_0^1 \left( 5r - r\sqrt{1+r^2} \right) \, dr \cdot \int_0^\pi \, d\theta\]
13.3 \( \int \int_R \frac{dA}{\sqrt{16-x^2-y^2}} \)

\( \mathbf{R} = \{(x,y) | x^2+y^2 \leq 4, x \geq 0, y \geq 0\} \)

\( \mathbf{R} = \{(r,\theta) | 0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}\} \)

\[
\int_0^\frac{\pi}{2} \int_0^2 r \cdot dr \cdot d\theta
\]

\[
= \int_0^\frac{\pi}{2} \int_0^2 r (16-r^2)^{-\frac{1}{2}} \cdot dr \cdot d\theta
\]

\[
= \int_0^\frac{\pi}{2} \int_0^2 16-r^2 \cdot dr \cdot d\theta
\]
\[ \iiint_0^1 e^{x^2} \, dx \, dy \, dz \]

Evaluate by first changing the order of integration.

Region of integration: \( 0 \leq y \leq 1, \) \( y \leq x \leq 1 \)

Redescribe region: \( 0 \leq x \leq 1, \) \( 0 \leq y \leq x \)
So \( \int \int \int \int e^{x^2} \, dy \, dx \) = 

**Inner:** \( \int_0^x e^{x^2} \, dy = e^{x^2}y \bigg|_{y=0}^{y=x} = xe^{x^2} - 0 \)

**Outer:** \( \int_0^1 xe^{x^2} \, dx = \left[ \frac{1}{2}e^{x^2} \right]_0^1 = \frac{1}{2}e - \frac{1}{2}e^0 = \frac{1}{2}(e-1) \)