12.8 (36) You do not have to do the second deriv. test. Just find the critical point, and argue that it yields a max volume by "common sense".

12.8 (60) I got \([1 \times 1 \times 10]\) (top & bottom are 1 by 1)

13.1 (48) I got \([8]\).
150 I got \([9]\).
42 I got \([\frac{20}{3}]\).

dist. from \((a,a^2)\) to \((b,b-1)\) is

\[D = \sqrt{(b-a)^2 + (b-1-a^2)^2},\]

so

\[D^2 = (b-a)^2 + (b-1-a^2)^2 = f(a,b),\]

\[f_a = 2(b-a)(-1) + 2(b-1-a^2)(-2a) \overset{\text{set}}{=} 0 \quad \text{(1)}\]

\[f_b = 2(b-a) + 2(b-1-a^2) \overset{\text{set}}{=} 0 \quad \text{(2)}\]
Note: \( 1 + 2 \Rightarrow 2(b - 1 - a^2)(1 - 2a) = 0 \)

So either \( a = \frac{1}{2} \) or \( b = a^2 + 1 \).

If \( a = \frac{1}{2} \), \( 2 \Rightarrow 2b - 1 + 2b - 2 - \frac{3}{2} = 0 \)

\[ \Rightarrow \quad 4b - \frac{7}{2} = 0 \quad \Rightarrow \quad b = \frac{7}{8} \]

\( 0 \Rightarrow -2b + 1 + \frac{7}{8} = 2b + 2 + \frac{3}{2} \)

\[ -4b + \frac{7}{2} = 0 \quad \Rightarrow \quad b = \frac{7}{8} \]

If \( b = a^2 + 1 \), \( 2 \Rightarrow 2(a^2 + 1 - a) = 0 \)

\[ \Rightarrow \quad a^2 - a + 1 = 0 \]

\[ a = \frac{1}{2} \quad \text{or} \quad b = \frac{7}{8} \]
\[ V = xyz = 10 \quad \leftarrow \text{constraint} \]

\[ \Rightarrow z = \frac{10}{xy} \]

\[ C_{97} = C = 2 \cdot 2x^2 + 1 \cdot 2y^2 + 10 \cdot 2xy \]

\[ = 2x^2 + 2y^2 + 20xy \]

\[ \Rightarrow C = \frac{20}{y} + \frac{20}{x} + 20xy = 20 \left[ \frac{1}{y} + \frac{1}{x} + xy \right] \]

\[ = f(x, y) \]

\[ f_x = C_x = 20 \left[ -\frac{1}{x^2} + y \right] \quad \text{set} \quad 0 \quad \text{1} \]

\[ C_y = 20 \left[ -\frac{1}{y^2} + x \right] \quad \text{set} \quad 0 \quad \text{2} \]
Substitution gives \(-\frac{1}{(\frac{1}{x^2})^2} + x = 0\)

\[ \Rightarrow -x^4 + x = 0 \]

\[ \Rightarrow -x(x^3 - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1. \]

\[ C_{xx} = 20 \left[ \frac{2}{x^3} \right], \quad C_{yy} = 20 \left[ \frac{2}{y^3} \right], \quad C_{xy} = 20 \left[ 1 \right] \]

so \( D(x, y) = \left( \frac{40}{x^3} \right) \left( \frac{40}{y^3} \right) - \left( 20 \right)^2 \)

so \( D(1, 1) = \left( \frac{40}{1^3} \right) \left( \frac{40}{1^3} \right) - \left( 20 \right)^2 = 0 \) \( \Rightarrow \) local \( \min \).

13.1 (19) \[ \int_{0}^{1} \int_{0}^{y} \frac{3y}{\sqrt{x+y^2}} \, dx \, dy \]

Inner:

\[ \int_{1}^{y} 3y (x+y^2)^{-\frac{1}{2}} \, dx \]

let \( u = x+y^2 \)

\[ = \int_{1+y^2}^{y} \frac{4+y^2}{1+y^2} \, du \]
\[ \begin{align*}
&= 3y \cdot 2u^{\frac{3}{2}} \bigg|_{u=4+y^2}^{u=1+y^2} = 6y \left(4+y^2\right)^{\frac{3}{2}} - 6y \left(1+y^2\right)^{\frac{3}{2}}. \\
\text{Outer: } &\int_0^1 (\text{above}) \, dy = \int_0^1 6y \left(4+y^2\right)^{\frac{3}{2}} \, dy - \int_0^1 6y \left(1+y^2\right)^{\frac{3}{2}} \, dy \\
&= \int_0^1 3 u^{\frac{3}{2}} \, du - \int_1^2 3 u^{\frac{3}{2}} \, du \\
&= \left. 3 \frac{2}{3} u^{\frac{3}{2}} \right|_0^1 - \left. 3 \frac{2}{3} u^{\frac{3}{2}} \right|_1^2 \\
&= \left. 3 \frac{2}{3} u^{\frac{3}{2}} \right|_0^5 - \left. 3 \frac{2}{3} u^{\frac{3}{2}} \right|_1^5 \\
&= 2 \left(5^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) - 2 \left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \\
&= 10\sqrt{5} - 16 - 4\sqrt{2} + 2 = 10\sqrt{5} - 14.
\end{align*} \]
13.2 Double integrals with non-rectangular region of integration

\[ \iiint_R f(x, y) \, dA \]

**EX:** Find volume under \( f(x, y) = 10 - \frac{1}{3} x^2 \) and above the region:

\[ y = \sqrt{x} \]

so the 3D region looks like:
\[ A(x) = \int_0^{\sqrt{x}} f(x,y) \, dy \]

Total volume = \[ \int_0^{4} \int_{y^2}^{\sqrt{x}} f(x,y) \, dy \, dx \]

\[ A(y) = \int_{y^2}^{4} f(x,y) \, dx \]

Total volume = \[ \int_0^{2} \int_{y^2}^{4} f(x,y) \, dx \, dy \]
The main potential trouble-water in 13.2 is the description of the region of integration:

\[ R = \{(x, y) \mid 0 \leq x \leq 4 \text{ and } 0 \leq y \leq \sqrt{x}\} \]

\[ y = \sqrt{x}, \quad x = y^2 \]
\[ R = \{ (x, y) \mid 0 \leq y \leq 2 \wedge y^2 \leq x \leq 4 \} \]

Another.

\[ R = \{ (x, y) \mid 0 \leq y \leq 1 \text{ and } 0 \leq y \leq x \} \cup \{ (x, y) \mid 1 \leq x \leq 3 \text{ and } 0 \leq y \leq \frac{3}{2} - \frac{1}{2}x \} \]

or

\[ R = \{ (x, y) \mid 0 \leq y \leq 1 \text{ and } y \leq x \leq 3 - 2y \} \]