Need a volume function:

\[ V = xyz = (96 - 2y - 2z)y^2z \]

\[ = 96yz - 2y^2z - 2yz^2 = f(y, z). \]

Now, use calculus:

\[ f_y = 96z - 4yz - 2z^2 \quad \frac{\text{set}}{0} \]

\[ f_z = 96y - 2y^2 - 4yz \quad \frac{\text{set}}{0}. \]
\[
\begin{align*}
2z(48 - 2y - z) &= 0 \quad \text{(1)} \\
2y(48 - y - 2z) &= 0 \quad \text{(2)}
\end{align*}
\]

(1) \Rightarrow \text{either } z \neq 0 \text{ or } 48 - 2y - z = 0

(2) \Rightarrow \text{either } y \neq 0 \text{ or } 48 - y - 2z = 0

\[
\begin{align*}
2y + z &= 48 \\
y + 2z &= 48
\end{align*}
\]

\Rightarrow \ y = 16, \ z = 16.

(\text{so } x = 96 - 2(16) - 2(16) = 32). \]

So only critical pt. is \( y = 16, \ z = 16 \) \( (x = 32) \).

\[
D = f_{yy} f_{zz} - f_{yz}^2 = (-4z)(-4y) - (96 - 4y - 4z)^2
\]

\[
D(16, 16) = (-64)(-64) - (-32)^2 = +, \ \text{local max}.
\]
Find pt. on plane \( x+y+z=4 \) closest to \((0,3,6)\).

To minimize the distance from some arbitrary point on the plane to \((0,3,6)\),

**Note:** to minimize a distance, you may minimize the square of the distance, then square-root the result.

The distance from \((x,y,z)\) to \((0,3,6)\) is

\[
D = \sqrt{(x-0)^2 + (y-3)^2 + (z-6)^2}
\]

So \(D^2 = x^2 + (y-3)^2 + (z-6)^2\).
constraint, since \((x, y, z)\) must be on \(x + y + z = 4\), I know \(z = 4 - x - y\), substitute this in \(D^2\):

\[
D^2 = x^2 + (y-3)^2 + (-x-y-2)^2 = f(x, y).
\]

\[
\frac{\partial f}{\partial x} = 2x + 0 + 2(-x-y-2)(-1)
\]

\[
= 2x + 2x + 2y + 4 = 4x + 2y + 4 \overset{set}{=} 0
\]

\[
\frac{\partial f}{\partial y} = 0 + 2(y-3) \cdot 1 + 2(-x-y-2)(-1)
\]

\[
= 2y - 6 + 2x + 2y + 4 = 2x + 4y - 2 \overset{set}{=} 0
\]

Solve, check that \(y = 0\).

This crit. pt. will clearly give minimize \(D^2\):
Last time:

\[ z = f(x, y) \]

Cross-sectional area

\[ A(y) = \int_a^b f(x, y) \, dx \]

Total vol. = \[ \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy \]
Could also do:

\[ z = f(x, y) \]

cross-sectional area

\[ A(x) = \int_c^d f(x, y) \, dy , \]

\[ \text{total vol.} = \int_a^b A(x) \, dx = \int_a^c \int_b^d f(x, y) \, dy \, dx \]

So:

\[ \int_a^b \int_c^d f(x, y) \, dy \, dx \quad \text{should} \quad \int_c^a \int_b^d f(x, y) \, dx \, dy , \]

and this equality is true when \( f \) is continuous on \([a, b] \times [c, d]\) (Fubini's Thm)
Both versions of the double integral above can be written as

\[ \iint_{[a,b] \times [c,d]} f(x,y) \, dA \]

\( a \) to differential area element.
Another interpretation:

\[ \iint_{[a,b] \times [c,d]} f(x,y) \, dA \] says to add up

\[ f(x,y) \, dA \] throughout the rectangle \( [a,b] \times [c,d] \).

Can also think of \( f(x,y) \) as the density of a plate occupying \( [a,b] \times [c,d] \), in which case

\[ dx \, dy = dA \]
\[
\iint_{[a,b] \times [c,d]} f(x,y) \, dA \quad \text{gives the total mass of the plate.}
\]

Note: If \( f(x,y) \) goes negative within \([a,b] \times [c,d]\):

\[
z = f(x,y)
\]

Then
\[
\iint_{[a,b] \times [c,d]} f(x,y) \, dA = \left( \text{volume above} \right)_{xy \text{ plane}} - \left( \text{vol. below} \right)_{xy \text{ plane}}
\]

For convenience, we often see:
\[ R = \{ (x,y) \mid a \leq x \leq b \text{ and } c \leq y \leq d \}, \]

then write \[ \iint_R f(x,y) \, dA \]

**Fact:** \[ \iint_R f(x,y) \, dA = \left( \text{avg. value of } f(x,y) \text{ on } R \right) \cdot \left( \text{area of } R \right) \]

13. (26) \[ R = \{ (x,y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq \frac{4}{3} \} = [0,1] \times [0,\frac{4}{3}] \]

Find \[ \iint_R y \cos(xy) \, dA. \]

I'll set it up as \[ \int_0^{\frac{4}{3}} \int_0^1 y \cos(xy) \, dx \, dy. \]
Inner: \[ \int_0^1 y \cos(xy) \, dx = y \cdot \frac{y}{2} \sin(yx) \bigg|_{x=0}^{x=1} \]

\[ = \sin(y) \cdot \frac{y}{2} \]

Outer: \[ \int_0^\frac{\pi}{3} \sin(y) \, dy = -\cos(y) \bigg|_0^{\frac{\pi}{3}} = -\cos\left(\frac{\pi}{3}\right) - (-\cos(0)) \]

\[ = -\frac{1}{2} + 1 = \frac{1}{2} \]