Math 202: Calculus for Business and Economics
Spring 2015 Exam 2 Version B

Directions:
• Read and follow all directions carefully.
• You must show all work or justify your reasoning with meaningful evidence to receive full credit, regardless of whether it is explicitly asked for in the question.
• Clearly indicate your final answer for each problem.
• No use of calculators, books, notes, cell phones, or other electronic devices allowed during the exam.

1. (4 points each) Find the derivative of each of the following functions. Do not simplify.
   a. \( f(x) = \frac{2}{x^3} + \frac{e^x}{3} - 4 \ln x = 2x^{-3} + \frac{1}{3} e^x - 4 \ln(x) \), so
      \[ f'(x) = -6x^{-4} + \frac{1}{3} e^x - 4 \cdot \frac{1}{x} \]
   b. \( h(x) = \ln(x^2 - 9x + 3) \)
      \[ h'(x) = \frac{1}{x^2 - 9x + 3} (2x - 9) \]
   c. \( f(x) = (x^3 + 5)e^{4x} \)
      \[ f'(x) = (x^3 + 5)e^{4x} + e^{4x} (3x^2) \]
   d. \( y = \frac{x^3}{8 - x^2} \)
      \[ y' = \frac{(8-x^2)(3x^2) - (x^3)(-2x)}{(8-x^2)^2} \]
2. (10 points) Use differential approximations in the following problem:
A company manufactures and sells $x$ televisions per month. Let revenue function be $R(x) = 400x - \frac{1}{50}x^2$. What is the approximate change in revenue if production is increased from $x_1 = 2,500$ to $x_2 = 2,510$? (Hint: for $y = f(x)$, $\Delta y \approx dy = f'(x)dx$). Label units in your solution.

\[
R'(x) = 400 - \frac{1}{25}x, \quad \text{so} \quad R'(2500) = 400 - \frac{2500}{25} = 300.
\]
\[\Delta x = 2510 - 2500 = 10, \quad \text{so} \]
\[\Delta R \approx dR = (300)(10) = \boxed{3000 \text{ dollars}}\]

3. (8 points) Use implicit differentiation to find $y'$ for the relation $x^2 + xy - y^3 = 2$.

Think: $x^2 + x(y) - (y)^3 = 2$

Take $\frac{d}{dx}$: $2x + x\cdot y' + y - 3y^2y' = 0$

Solve for $y'$:

\[x y' - 3y^2 y' = -2x - y\]
\[y'(x - 3y^2) = -2x - y\]
\[y' = \frac{-2x - y}{x - 3y^2}\]
4. (10 points) Do ONE of the following two problems. If you have time to work on both problems, only the better effort will count. Simplify your solution completely.

A 10 ft ladder is placed against a vertical wall. Suppose that the top of the ladder is sliding down the wall at a constant rate of 3 ft/sec, how fast is the bottom of the ladder sliding away from the wall when the bottom of the ladder is 6 ft away from the wall? (Hint: The Pythagorean theorem is $a^2 + b^2 = c^2$). Label units in your solution.

The volume ($V$) and the radius ($r$) of a spherical balloon are related by the formula, $V = \frac{4}{3} \pi r^3$.

At a certain moment, the balloon's radius is 20 cm, and is growing at a rate of $\frac{dr}{dt} = 2 \text{ cm/min}$. At what rate is the volume of the balloon growing at that same moment? Label units in your solution.

**Rate relation:**

$$\frac{dV}{dt} = \frac{4}{3} \pi r^2 \frac{dr}{dt}$$

or

$$\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}.$$  

At the MoI:

$$\frac{dV}{dt} = 4 \pi (20)^2 \left( \frac{2}{2} \right)$$

$$= 3200 \pi \text{ cm}^3/\text{min}$$

**MOI:**

\[ \begin{align*}
\text{MOI:} & \quad 3 \text{ft} \\
\text{at} & \quad 10 \\
\text{and} & \quad 6 \\
\text{General:} & \quad \frac{dy}{dt} = -3 \\
\text{Equations:} & \quad x^2 + y^2 = 100 \\
& \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \\
& \quad 2x \frac{dx}{dt} + y \frac{dy}{dt} = 0. \\
\text{At MOI,} & \quad 6 \frac{dx}{dt} + 8(-3) = 0 \\
\Rightarrow & \quad \frac{dx}{dt} = 4 \text{ ft/sec}
\end{align*} \]
5. (10 points) Let \( x = f(p) = 600 - 10p^2 \) be the price - demand equation. Determine whether the demand is elastic, is inelastic, or has unit elasticity at the price, \( p = 5 \). Circle your solution. Show your work. The formula for the elasticity of demand is:

\[
E(p) = \frac{-p f'(p)}{f(p)} = \frac{-p (-20p)}{600 - 10p^2} = \frac{20p^2}{600 - 10p^2}
\]

\[
E(5) = \frac{2(5)^2}{60 - (5)^2} = \frac{50}{35} = \frac{10}{7} > 1
\]

**ELASTIC DEMAND**

**INELASTIC DEMAND**

**UNIT ELASTICITY**

6. (4 points each) Find each limit. Use L’Hospital’s Rule if it applies and justify why it is used. If L’Hospital’s Rule does not apply, find the limit by other means.

a. 
\[
\lim_{{x \to -3}} \frac{x^2}{(x + 3)^4} = \infty
\]

b. 
\[
\lim_{{x \to \infty}} \frac{5x + 3}{e^x} = \lim_{{x \to \infty}} \left( \frac{5}{e^x} \right) = \frac{5}{\infty} = 0
\]

c. 
\[
\lim_{{x \to 0}} \frac{\ln(1 + 3x)}{x} = \lim_{{x \to 0}} \left( \frac{3}{1 + 3x} \right) = \frac{3}{1} = 3
\]
7. Use the function, \( f(x) = x^4 e^x \) to answer the following questions.

a. (4 points) Find \( f'(x) \) and show that it can be factored as \( f'(x) = x^3(4 + x)e^x \)

\[
f'(x) = x^4 e^x + 4x^3 e^x = x^3 e^x (x + 4) .
\]

b. (10 points) Use an analysis of \( f'(x) = x^3(4 + x)e^x \) to fill in the blanks in the table below. Write NONE if necessary.

\[
\begin{array}{c|ccc}
 x & f''(x) & + & - \\
\hline
 -4 & - & + \\
0 & + & - & + \\
\end{array}
\]

All intervals on which \( f(x) \) is increasing: 
\((-\infty, -4)\), \((0, \infty)\)

All intervals on which \( f(x) \) is decreasing: 
\((-4, 0)\)

All \( x \) values where \( f(x) \) has a local maximum: 
\( x = -4 \)

All \( x \) values where \( f(x) \) has a local minimum: 
\( x = 0 \)
8. (10 points) Find all intervals on which the function \( g(x) = x^4 - 6x^2 \) is concave upward or concave downward and all inflection points of \( f(x) \). Write NONE if necessary.

\[
g'(x) = 4x^3 - 12x
\]

\[
g''(x) = 12x^2 - 12 = 12(x^2 - 1)
\]

\[
= 12(x-1)(x+1);
\]

<table>
<thead>
<tr>
<th>( g''(x) )</th>
<th>+</th>
<th>0</th>
<th>-</th>
<th>+</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All intervals on which \( f(x) \) is concave upward: \( (1, \infty), (-\infty, -1) \)

All intervals on which \( f(x) \) is concave downward: \( (-1, 1) \)

All \( x \) values where \( f(x) \) has an inflection point: \( x = 1, x = -1 \)
9. (10 points) Use the information given below about $f(x)$ to sketch the graph of $y = f(x)$. **Label** any local extrema and inflection points using ordered pairs and also label any vertical or horizontal asymptotes.

$f(x)$ is continuous on its entire domain, $(-\infty, \infty)$

\[
\lim_{x \to -\infty} f(x) = 6 \\
\lim_{x \to \infty} f(x) = \infty
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

$f'(x)$

\[
\begin{array}{ccc}
& & \\
\hline
& & \\
0 & \downarrow & +
\end{array}
\]

$x = 1$

$f''(x)$

\[
\begin{array}{ccc}
& & \\
\hline
& & \\
+ & \downarrow & 0
\end{array}
\]

$x = 3$

[Diagram with labels for increasing/decreasing, concavity, asymptotes, local minima, and inflection points.]