SHOW APPROPRIATE WORK or EXPLANATION on each problem for full credit. Box or circle your final answers. Calculators/note sheets are NOT allowed. Numbers in the [] indicate what each problem is worth. Check the board for useful info.

\[
\sin(2x) = 2 \sin(x) \cos(x) \quad \sin^2(x) = \frac{1}{2} [1 - \cos(2x)] \quad \cos^2(x) = \frac{1}{2} [1 + \cos(2x)]
\]

\[
1 - \sin^2(x) = \cos^2(x) \quad \tan^2(x) + 1 = \sec^2(x) \quad \sec^2(x) - 1 = \tan^2(x)
\]

\[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C
\]

\[
\int \sec^3(x) \, dx = \frac{1}{2} \left( \sec(x) \tan(x) + \ln | \sec(x) + \tan(x) | \right) + C
\]

\[
\int \tan(x) \, dx = \ln | \sec(x) | + C
\]

\[
\int \csc(x) \, dx = -\ln | \csc(x) + \cot(x) | + C
\]

\[
\int \cot(x) \, dx = \ln | \sin(x) | + C
\]

\[
\int \csc^3(x) \, dx = \frac{1}{2} \left( -\csc(x) \cot(x) + \ln | \csc(x) - \cot(x) | \right) + C
\]

\[
\int \sec(x) \, dx = \ln | \sec(x) + \tan(x) | + C
\]

\[
\int \ln(x) \, dx = x \ln(x) - x + C
\]

1. [12 each] Do each of the following integrals (fractions in your answers can be left un-combined):

a) \[ \int \frac{1}{x^2\sqrt{x^2 - 4}} \, dx \]

Let \( x = 2 \sec \theta \). Then \( dx = 2 \sec \theta \tan \theta \, d\theta \),

\( \sqrt{x^2 - 4} \) becomes \( 2 \tan \theta \) and also \( \sec \theta = \frac{x}{2} \):

\[
\int \frac{1}{4 \sec^2 \theta \cdot 2 \tan \theta} \, d\theta = \frac{1}{4} \int \cos \theta \, d\theta
\]

\[
= \frac{1}{4} \sin(\theta) + C = \frac{1}{4} \left( \frac{\sqrt{x^2 - 4}}{x} \right) + C
\]
\[ b) \int \frac{(2x + 5) \sin(3x)}{(x + 3)(x^2 + 5)} \, dx = \frac{(2x + 5)(-\frac{1}{3} \cos(3x))}{9} - \int \frac{\cos(7x)}{3} \cdot 2 \, dx \]
\[ = \frac{-(2x + 5)}{3} \cos(3x) + \frac{2}{3} \int \cos(3x) \, dx \]
\[ = -\frac{1}{3} (2x + 5) \cos(3x) + \frac{2}{3} \sin(3x) + C \]

\[ c) \int \frac{2x + 2}{(x + 3)(x^2 + 5)} \, dx = A \frac{2x + 2}{x + 3} + B \frac{x + 5}{x^2 + 5} = \frac{A(x^2 + 5) + (Bx + C)(x + 3)}{(x + 3)(x^2 + 5)} \]

Set \( x = -3 \) \[ \Box^3 \), \( y + 9 = 14 \Rightarrow A = -\frac{2}{7}. \]

Set \( x = 0 \) \[ \Box^5 \), \( 2 = 5A + 3C \Rightarrow C = \frac{2 - 5(-\frac{2}{7})}{3} = \frac{8}{7}. \]

\[ x^2 \text{ terms in } \Box^5: \quad 0x^2 = A(x^2 + Bx^2) \Rightarrow B = -\frac{3}{7}. \]

\[ = \int \left( -\frac{2}{7} \frac{x + 5}{x + 3} + \frac{2}{7} \frac{x + 5}{x^2 + 5} \right) \, dx = -\frac{2}{7} \ln|x + 3| + \frac{1}{7} \int \frac{2x}{x^2 + 5} \, dx + \frac{8}{7} \int \frac{1}{x^2 + 5} \, dx \]
\[ = -\frac{2}{7} \ln|x + 3| + \frac{1}{7} \ln(x^2 + 5) + \frac{8}{7\sqrt{5}} \tan^{-1} \left( \frac{x}{\sqrt{5}} \right) + C \]
1. (continued)

d) \[ \int_1^\infty \frac{x + 5}{x^3} \, dx = \int_1^t (x^{-2} + 5x^{-3}) \, dx = \left[ -\frac{1}{x} - \frac{5}{2x^2} \right]_1^t = \left[ -\frac{1}{t} - \frac{5}{2t^2} \right] + \left[ 1 + \frac{5}{2} \right] \]

\[ = -\frac{3}{2} - \frac{1}{t} - \frac{5}{2t^2}, \text{ which \to } \frac{3}{2} \text{ as } t \to \infty. \quad (\text{convergent}) \]

2. [10] Sketch the region enclosed by the curves \( y = 1/x, \ y = x^2, \) and \( y = 4 \). Then find the area of this region.

![Graph of the region](image)

\[ \text{Area} = \int_1^4 (\sqrt{y} - \frac{1}{y}) \, dy = \left[ \frac{2}{3} y^{\frac{3}{2}} - 2\ln(y) \right]_1^4 = \left[ \frac{16}{3} - 2\ln(4) \right]. \]

\[ \text{OR:} \quad \text{Area} = \int_1^1 (\sqrt{y} - \frac{1}{y}) \, dy + \int_1^2 (1 - x^2) \, dx = \ldots = \text{above.} \]
3. Let $D$ be the region bounded by the curves $y = x^3$ and $x = y^2$, as shown.

a) [10] Sketch the solid formed when $D$ is rotated about the $y$-axis. Then SET UP an integral or integrals for finding its volume.

- Using shells:
  $$\int_0^1 2\pi x (\sqrt{x} - x^3) \, dx$$

- Using washers:
  $$\int_0^1 \pi (y^2) - \pi (y^3) \, dy$$

b) [10] Sketch the solid formed when $D$ is rotated about the horizontal line $y = 2$. Then SET UP an integral or integrals for finding its volume.

- Washers:
  $$\int_0^1 \pi (2 - x^3)^2 - \pi (2 - \sqrt{x})^2 \, dx$$

- Shells:
  $$\int_0^2 2\pi (2-y) (2y - y^2) \, dy$$
4. The piece of the curve \( x = e^y \) from \((1,0)\) to \((e,1)\) is rotated about the \(x\)-axis to form a surface. Set up an integral which would give the area of this surface.

\[
\text{Surface Area} = \int_1^e 2\pi y(x) \sqrt{1 + (e^y)^2} \, dx = \int_0^1 2\pi x \sqrt{1 + (e^x)^2} \, dy
\]

5. [2 each] Short answer. Though it is not required, explanation/work could earn partial credit.

a) To do \( \int \frac{\sqrt{x^2 + 5}}{x} \, dx \), one should use the trigonometric substitution \( x = \sqrt{5} \tan \theta \).

b) To do \( \int \sin^3(x) \cos^4(x) \, dx \), one should use the substitution \( u = \cos(x) \).

c) Complete the following definite integral for finding the length of the curve \( y = \sin(x) \), \(0 \leq x \leq \pi\):

\[
\int_0^\pi \sqrt{1 + \cos^2(x)} \, dx
\]

d) Is \( \int_2^\infty \frac{x^2}{10x^3 - 1} \, dx \) convergent or divergent?

\( \text{Convergent} \)

e) Explain in one short sentence why \( \int_1^4 \frac{1}{\sqrt{x-2}} \, dx \) is improper.

\( \text{VA at } x = 2, \text{ which is } 1 \leq x \leq 4. \)

f) Give the form of the partial fraction decomposition for \( \frac{x^4 + 2}{(x-2)^3(x^2 + x + 2)} \).

\[
\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx + E}{x^2 + x + 2}
\]