On problems which say “no work necessary”, showing work is not required in order to receive full credit, but you may still receive partial credit on these problems if you do show work or explanation. You must show appropriate work or explanation on all other problems for full credit. Box or circle your final answers. Calculators/note sheets are NOT allowed. Numbers in the [] indicate what each problem is worth. Filling in your name, ID and section is worth 6 points. Check the board; there may be useful info up there. Hope you enjoy the winter break. Potentially useful formulas:

\[
\begin{align*}
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2(x) &= \frac{1}{2}(1 + \cos(2x))
\end{align*}
\]

For a sphere: Volume = \(\frac{4}{3}\pi(radius)^3\), Surface area = \(4\pi(radius)^2\)
For a circle: Area = \(\pi(radius)^2\), Circumference = \(2\pi(radius)\)
For a triangle: Area = \(\frac{1}{2}(base)(height)\)
For a rectangle: Area = \(base)(height)\)

1. [10] Sketch the graph of a function \(f\) that satisfies all of the conditions listed below.

- \(f\) is continuous on \((-\infty, \infty)\)
- \(\lim_{x \to -\infty} (f(x)) = 4\)
- \(\lim_{x \to -\infty} (f(x)) = \infty\)
- \(f(3) = 1\), \(f'(3)\) is undefined
- \(f'(x) > 0\) for \(x > 3\)
- \(f'(x) < 0\) for \(x < 3\)
2. [7 each] Find each limit. Use \( \infty \) or \(-\infty\) as an answer if appropriate. If you use L'Hospital's Rule, you must indicate how you know that the limit is indeterminate.

a) \( \lim_{{x \to 4}} \frac{x^3 - 4x^2}{x^2 + x - 20} \)

b) \( \lim_{{x \to 3^+}} \frac{1 - x}{x^2 - 3x} \)

c) \( \lim_{{x \to \infty}} \frac{9 + 6x^2 + x^4}{30x^2 - 3x^5} \)

d) \( \lim_{{x \to 0}} \frac{\sin(x)}{1 - e^{3x}} \)
3. [7 each] Find \( \frac{dy}{dx} \) for each function. Algebraic simplification is not necessary.

a) \( y = \frac{2}{3\sqrt{x}} + \frac{4}{x^3} \)

b) \( y = x^3 e^{5x} \)

c) \( y = \ln(3 + \arctan(x)) \)

d) \( y = \frac{\cos(3x)}{(x^2 + 2)^5} \)
4. [10] For \( f(x) = \frac{x - 1}{2x^2} \), show that the derivative is \( f'(x) = \frac{-x + 2}{2x^3} \), and then answer the questions below by analyzing the sign of \( f'(x) \).

- interval(s) where \( f \) is increasing: 
- \( x \) value(s) where \( f \) has local maximum: 

5. [10] Find all intervals on which the function \( f(x) = xe^x \) is concave up.
6. [10] Below is a sketch of \( y = f(x) = \frac{1}{2}x^2 - 2 \) for \( 0 \leq x \leq 3 \). Find \( \int_0^3 \left( \frac{1}{2}x^2 - 2 \right) \, dx \), and explain what the value of this integral represents in the picture.

7. [10] Solve one of the following two problems. If you work on both, the better effort will be counted.

| Suppose oil spills from a ruptured tanker and spreads in a circular pattern. At the present, the radius of the spill is 600 meters and is increasing at 0.5 meters per second. At what rate is the area of the spill increasing? (\textit{hint}: related rates) | Find the slope of the line which is tangent to the curve \( x^3 y^3 - xy = 6 \) at the point (2, 1). (\textit{hint}: implicit differentiation) |
8. [10] Solve one of the following two problems. If you work on both, you will be credited with the better effort. You may leave answers “messy”, but label them with the correct units.

<table>
<thead>
<tr>
<th>A farmer has 800 feet of fencing, and he wants to build a rectangular pen with three sections, as shown below. He is using an existing fence for the back of the pen, so he only needs to use his 800 feet of fence on the parts that are drawn squiggly in the picture. What dimensions ((x, y)) should the farmer use in order to maximize the total enclosed area of the three-section pen?</th>
</tr>
</thead>
<tbody>
<tr>
<td>An open-top box is to be made with a base that is (x) inches by (2x) inches, and with height (y). If the volume of the box is to be 500 cubic inches, what dimensions ((x, y)) should the box have in order to minimize the amount of material used?</td>
</tr>
</tbody>
</table>

![Diagram of a rectangular pen with three sections and a diagram of an open-top box.](Attachment)
9. [8 each] Do any three of the following integrals. Numerical answers should be simplified as much as possible. If you try all 4, your best three will be counted.

a) \[ \int \left( 5x - \frac{3}{x^3} \right) \, dx \]

b) \[ \int_{1}^{2} \left( 2 - \frac{5}{t} \right) \, dt \]

c) \[ \int \left( \frac{e^x}{6} + \sqrt{x} \right) \, dx \]

d) \[ \int_{0}^{\frac{\pi}{2}} (x - 3 \cos(x)) \, dx \]
10. [8 each] Do any four of the following integrals. Numerical answers should be simplified as much as possible. If you try all 6, your best four will be counted.

a) \[ \int x^4 (2x^5 + 7)^8 \, dx \]

b) \[ \int \frac{1}{\sqrt{1 + 7x}} \, dx \]

c) \[ \int_0^{2\pi} x^2 \sin(x^3) \, dx \]
11. (continued)

d) $\int_{0}^{2} \frac{y^3}{2y^4 + 1} \, dy$

e) $\int \sin(x) \cos^2(x) \, dx$

f) $\int_{1}^{e} \frac{[\ln(x)]^2}{x} \, dx$
12. Let \( g(x) = \int_0^x f(t) \, dt \), where \( f \) is the function shown below. Answer the following (no work necessary):

a) [2] Evaluate \( g(6) \).

b) [2] On what subinterval of \([0, 8]\) is \( g \) decreasing?

c) [6] On the same axes as \( f \), draw the graph of \( g \) on \([0, 8]\) as precisely as you can.

13. [2 each] Fill in the blanks to correctly complete the following statements (no work necessary):

A) If \( \int_1^5 f(x) \, dx = 7 \) and \( \int_3^5 f(x) \, dx = 4 \), then \( \int_1^3 f(x) \, dx = \) 

B) Complete the statement of the Mean Value Theorem: If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a number \( c \) in \((a, b)\) such that 

C) Complete the definition of the derivative: \( f'(x) = \) 

D) If \( g(x) = \int_1^x \frac{1}{\sqrt{t} + 1} \, dt \), then \( g'(x) = \) 

E) If \( g(x) = \int_1^{x^3} \frac{1}{\sqrt{t} + 1} \, dt \), then \( g'(x) = \) 

F) The average value of the function \( f(x) = x^3 \) on the interval \([0, 2]\) is \( \)