1.4: Squeeze Thm

If \( f(x) \geq g(x) \geq h(x) \) for all \( x \rightarrow a \), then

\[
\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x),
\]

and

\[
\lim_{x \to a} f(x) = L.
\]
\[ \lim_{x \to 0} \sin (\frac{1}{x}) \]

\[ y = \sin (\frac{x}{4}) \]

\[ \lim_{x \to 0} \frac{\sin (x^2)}{(x^2)} \]

Find \( \lim_{x \to 0} \frac{x^2 \sin (\frac{1}{x})}{x} \). 

Ex. of using squeeze theorem.
\[ y = x^2 \sin (4x) \]
Proof that \( \lim_{x \to 0} (x^2 \sin(\frac{1}{x})) = 0 \):

For all \( x \neq 0 \),

\[-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2,\]

\[\frac{-x^2}{x} \leq \sin(\frac{1}{x}) \leq \frac{x^2}{x} = x.\]

(Could do: \(-1 \leq \sin(\frac{1}{x}) \leq 1\)

\[x^2: -x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2\]

And \( \lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} (x^2) \).

\( \lim_{x \to 0} (x^2 \sin(\frac{1}{x})) \) must \( = 0 \) by squeeze.
\[ \lim_{\theta \to 0} \left( \frac{\theta}{\sin \theta} \right) = 1 \]

\[ \lim_{\theta \to 0} \left( \frac{\theta}{\cos \theta} \right) = 1 \]

Also:

\[ \lim_{\theta \to 0} \frac{\theta}{\theta} = 1 \]
$$\lim_{\theta \to 0} \left( \frac{\theta}{\sin \theta} \right) = 3 \cdot 1 = 3$$

\[ \lim_{x \to 0} \left( \frac{x}{\sin 3x} \right) = \lim_{x \to 0} \left( \frac{3x}{\sin 3x} \cdot \frac{1}{3} \right) = \lim_{x \to 0} \left( \frac{3x}{\sin 3x} \right) = 1 \cdot \frac{3}{3} = 1 \]
So: \( \lim_{x \to a} x^n = a^n \)

\( \lim_{x \to a} (x^n) = a^n \)

(\( n = \text{integer} \))

Note: Any Thm which holds for 2-sided limits should hold for 1-sided limits as well.
This function is continuous only at $x=0$.

$f(x) = \begin{cases} 
1 & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}$

$x$ is rational if $x \neq 0$.
A pathological example:

\[
\lim_{x \to a} f(x) = f(a).
\]

A function \( f \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \).
\[
\lim_{x \to a^-} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a^+} f(x) = f(a). \]

If \( f \) is continuous from the right at \( a \), then

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x). \]
At what x-values is $f$ not continuous? All x except 1. Is $f$ left-continuous at 2?

At what x-values is $f$ continuous? All x except 1. Is $f$ right-continuous at 1? Yes.