Section 9.2: Power Series

Recall the $n^{\text{th}}$-degree Taylor Polynomial centered at $a$:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k$$

There’s a handy recipe for building $p_n(x)$. First, note that we can rewrite $p_n(x)$ as:

$$p_n(x) = \sum_{k=0}^{n} f^{(k)}(a) \cdot \frac{1}{k!} \cdot (x - a)^k$$
Form a table:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f^{(k)}(x)$</th>
<th>$f^{(k)}(a) \times \frac{1}{k!}$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>$\times$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$\times$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\times$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\times$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$\times$</td>
<td>$\frac{1}{24}$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$\times$</td>
<td>$\frac{1}{120}$</td>
</tr>
</tbody>
</table>
**Example**: Find the $n^{th}$-degree Taylor Polynomial for $f(x) = \ln x$ centered at $x = 1$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$a =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$f^{(k)}(x)$</td>
</tr>
<tr>
<td>0</td>
<td>$\times$</td>
</tr>
<tr>
<td>1</td>
<td>$\times$</td>
</tr>
<tr>
<td>2</td>
<td>$\times$</td>
</tr>
<tr>
<td>3</td>
<td>$\times$</td>
</tr>
<tr>
<td>4</td>
<td>$\times$</td>
</tr>
<tr>
<td>5</td>
<td>$\times$</td>
</tr>
</tbody>
</table>
\[ f(x) = \ln x \quad \quad a = 1 \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(a) )</th>
<th>( \times )</th>
<th>( 1/k! )</th>
<th>( = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \ln x )</td>
<td>0</td>
<td>( \times )</td>
<td>( 1/0! )</td>
<td>( = )</td>
</tr>
<tr>
<td>1</td>
<td>( x^{-1} )</td>
<td>1</td>
<td>( \times )</td>
<td>( 1/1! )</td>
<td>( = )</td>
</tr>
<tr>
<td>2</td>
<td>( -x^{-2} )</td>
<td>-1</td>
<td>( \times )</td>
<td>( 1/2! )</td>
<td>( = )</td>
</tr>
<tr>
<td>3</td>
<td>( 2x^{-3} )</td>
<td>2</td>
<td>( \times )</td>
<td>( 1/3! )</td>
<td>( = )</td>
</tr>
<tr>
<td>4</td>
<td>( -6x^{-4} )</td>
<td>-6</td>
<td>( \times )</td>
<td>( 1/4! )</td>
<td>( = )</td>
</tr>
<tr>
<td>5</td>
<td>( 24x^{-5} )</td>
<td>24</td>
<td>( \times )</td>
<td>( 1/5! )</td>
<td>( = )</td>
</tr>
</tbody>
</table>

\[ p_n(x) = \]
We’ve seen that

\[
\ln x \approx \sum_{k=1}^{n} (-1)^{k+1} \frac{(x - 1)^k}{k}.
\]

More precisely, at least for some values close to \( x = 1 \),

\[
\ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x - 1)^k}{k}.
\]
\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x - 1)^k}{k}
\]

is a power series.

The form of a power series is:
\[
\sum_{k=0}^{\infty} c_k (x - a)^k
\]

The important question will be: for what values of \( x \) will a power series converge?
Notice we don’t ask if there are *any* values of $x$ for which a power series will converge.

There’s at least one easy value of $x$ for which

$$\ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x - 1)^k}{k}.$$
In general, it makes sense that a power series

$$\sum_{k=0}^{\infty} c_k (x - a)^k$$

will always converge at $x =$

There may be other values near this for which the power series is also convergent. We’ll learn to find them soon. For now …
It turns out that \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} \) converges to \( \ln x \) for \( x \in (0,2] \).

This interval is the **interval of convergence**.

Notice that the value \( x = 1 \) is in the center of the interval. This is why \( x = 1 \) is called the **center** of this series.

The values \( x = 0 \) and \( x = 2 \) are the endpoints of the interval of convergence. The distance from the center to each endpoint is called the **radius of convergence**.
Properties of Power Series

A very easy power series is:

\[ \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \ldots \]

This power series is centered at \( a = \) 

This series is _________ and thus converges when \( x \)
More about this series:

\[ \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots \]

The interval of convergence is \( I = \)

The radius of convergence is

When \( x \in I \), the series converges to
We’ve seen that

\[ \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } x \in (-1, 1) \]

In general, we will write

\[ f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \text{ for } x \in I \]

where \( I \) is an interval centered at \( a \).
Finding the Interval of Convergence

We’ll use the ratio test. For a power series \( \sum_{k=0}^{\infty} c_k(x - a)^k \), compute the ratio
\[
\frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k} = \frac{c_{k+1}}{c_k} (x - a).
\]

Then find the limit of that:
\[
r = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} (x - a) \right|.
\]

According to the ratio test, the series converges when \( 0 \leq r < 1 \). So solve the equation \( r = 1 \).
This boils down to finding \( c = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right| \) and then solving

\[
(x - a)c = 1 \quad \text{and} \quad (x - a)c = -1
\]

for \( x \).

So the endpoints of the interval of convergence are

\[
x = a \pm \frac{1}{c}
\]
Notes:

- If $c = 0$ the series converges everywhere, and the radius of convergence is

- If $c = \infty$ the series converges only at $x =$ and the radius of convergence is

- We can’t use this to determine convergence at the endpoints of the interval. (Why?) We’ll have to check those separately.
Example: Find the radius of convergence and interval of convergence for

\[ f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x - 2)^k}{4^k} \]

Center: \( x = \)

\( c_k = \)
Since $c_k = \frac{(-1)^k}{4^k}$,

$$c = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right| =$$

Endpoints: $x = a \pm \frac{1}{c}$
Interval of convergence:

Test the endpoints.
\[ f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x - 2)^k}{4^k} \]

\[ x = -2: \]
$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x - 2)^k}{4^k}$$

$x = 6$: 
Example: Find the radius of convergence and interval of convergence for

\[ f(x) = \sum_{k=1}^{\infty} \frac{(x + 3)^k}{\sqrt{k} \cdot 2^k} \]

Center: \( x = \) 

\( c_k = \)
Since $C_k = \frac{1}{\sqrt{k} \cdot 2^k}$,

c = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right| =

Endpoints: $x = a \pm \frac{1}{c}$
\[ f(x) = \sum_{k=1}^{\infty} \frac{(x + 3)^k}{\sqrt{k} \cdot 2^k} \]
\[ f(x) = \sum_{k=1}^{\infty} \frac{(x + 3)^k}{\sqrt{k} \cdot 2^k} \]

\[ x = -1: \]
Example: Find the radius of convergence and interval of convergence for \( f(x) = \frac{1}{5-x} \).

This looks a bit like a geometric series result, except that \( \frac{1}{1-x} \) would be rewritten: 

\[ f(x) = \frac{1}{5-x} \]
\[
f(x) = \frac{1}{5} \sum_{k=0}^{\infty} \frac{x^k}{5^k}
\]

Center: \( x = \)

Interval of Convergence:

Radius of Convergence:
Note: \( \frac{1}{5-x} = \frac{1}{5} + \frac{x}{5^2} + \frac{x^2}{5^3} + \cdots = \sum_{k=0}^\infty \frac{x^k}{5^{k+1}} \)

**Example:** Find a power series representation for \( f(x) = \ln(5 - x) \).
\[ f(x) = -\int \sum_{k=0}^{\infty} \frac{x^k}{5^{k+1}} \, dx = \]