Section 8.1: Sequences and Infinite Series

A sequence is a list of numbers written in some specific order:

3, 6, 12, 24, ...

Notation: \( \{ a_n \}_{n=1}^{\infty} = \{ a_n \} = \{ 3, 6, 12, 24, \ldots \} \)
We can write \( \{ a_n \} = \{ 3, 6, 12, 24, \ldots \} \) implicitly as a recurrence relation:

\[
a_1 = 3, \text{ and } a_{n+1} = 2a_n \quad \text{for } n = 1, 2, 3, \ldots
\]

Or, we can write \( \{ a_n \} = \{ 3, 6, 12, 24, \ldots \} \) explicitly:

\[
a_n = 3 \times 2^{n-1} \quad \text{for } n = 1, 2, 3, \ldots
\]
We’ll be interested in limits of sequences, if they exist.

\[ a_n = 3 \times 2^{n-1} \]  for \( n = 1, 2, 3, \ldots \) diverges.
What about $a_n = \frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$?

The terms approach:
Example: Find the first four terms of the sequence
\[ a_n = (-1)^n \] for \( n = 1, 2, 3, \ldots \) and determine if it is convergent or divergent.
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\[ a_n = \frac{(-1)^n}{n^2} \] for \( n = 1, 2, 3, \ldots \) and determine if it is convergent or divergent.
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Infinite Series

Consider the finite number $0.333333\ldots = 0.\overline{3}$.

This can be written a sum of infinitely many terms:

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$$
If we stop at any point, that’s a **partial sum**:

\[ S_1 = 0.3 \]
\[ S_2 = 0.3 + 0.03 \]
\[ S_3 = 0.3 + 0.03 + 0.003 \]

\[ S_n = 0.3 + 0.03 + 0.003 + \ldots = \sum_{k=1}^{n} (3 \times 0.1^k) \]

\[ n \text{ of these} \]
The sum of the terms in sequence of partial sums is an **infinite series**.

If the sequence of the partial sums has a limit \( L \), the series converges to that limit \( L \).

For our earlier sequence of partial sums \( \sum_{k=1}^{n} (3 \times 0.1^k) \),

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} (3 \times 0.1^k) = \sum_{k=1}^{\infty} (3 \times 0.1^k) = \]

Example: Does the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge or diverge?

$$\sum_{k=1}^{\infty} \frac{1}{k^2} =$$
Note: recall that $\int_{1}^{\infty} \frac{1}{x^2} \, dx$ converges to 1.

Riemann sums: