Using Common Random Numbers: for some systems, there are two competing methods $g$ and $h$ for completing some task, with respective means $\theta_1 = E[g(T)]$, $\theta_2 = E[h(T)]$, and $T = (T_1, \ldots, T_n) \sim F$, for some distribution $F$; simulation is needed to determine if $\theta = \theta_1 - \theta_2$.

- Example: $T_i$s are times for completing jobs by two machines; method $g$ (or $h$) completes longest (or shortest) jobs first.

- Question: should simulation use
  $\theta = E[g(T) - h(T^*)] \approx \frac{1}{N} \sum_{i=1}^{N} (g(T_i) - h(T^*_i))$
  with $T_i \sim F$, $T^*_i \sim F$, or use “common” RNs
  $\theta = E[g(T) - h(T^*)] \approx \frac{1}{N} \sum_{i=1}^{N} (g(T_i) - h(T_i))$?

- Analysis

  \[
  \text{Var}(g(T) - h(T^*)) = \text{Var}(g(T)) + \text{Var}(h(T^*)) - 2\text{Cov}(g(T), h(T^*)) \]

  If $g(T)$ and $h(T)$ are positively correlated, use common RNs;
  If $g(T)$ and $h(T)$ are increasing, then positively correlated.
• Machine Example: \( n = 20 \), Exp(1) service times, Matlab

\[
N = 1000; \\
\text{for } i = 1:N, T = -\log(\text{rand}(1,20)); \\
\hspace{1cm} t(i) = \text{prllsvs}(T,0) - \text{prllsvs}(T,1); \\
\text{end, disp([mean(t) var(t)])} \\
\hspace{1cm} 5.5312 \quad 3.2459 \\
\text{for } i = 1:N, T = -\log(\text{rand}(1,20)); \\
\hspace{1cm} Ts = -\log(\text{rand}(1,20)); \\
\hspace{1cm} t(i) = \text{prllsvs}(T,0) - \text{prllsvs}(Ts,1); \\
\text{end, disp([mean(t) var(t)])} \\
\hspace{1cm} 5.517 \quad 11.987
Barrier Option

- Stock price $P(t)$ has lognormal distribution:
  $P(t) = ve^{X}$, with $X \sim Normal(\mu t, \sigma^2 t)$, $P(0) = v$.
- Standard European call option gives right to buy at time $t$ for price $K$.
- Expected value of the option $C(K, t, v)$ is

$$C(K, t, v) = E[(P(t) - K)^+]$$

$$= \int_{-\infty}^{\infty} (ve^{x} - K)^+ \frac{e^{-(x-\mu t)^2/(2\sigma^2)}}{\sigma \sqrt{2\pi}} dx$$

$$= \int_{\ln(K/v)}^{\infty} (ve^{x} - K) \frac{e^{-(x-\mu t)^2/(2\sigma^2)}}{\sigma \sqrt{2\pi}} dx,$$

with computation from formula:

$$C(K, t, v) = ve^{t\mu + t\sigma^2/2} \left( 1 - \Phi\left(\frac{\ln(K/v) - t\mu - t\sigma^2}{\sqrt{t\sigma}}\right) \right)$$

$$- K \left( 1 - \Phi\left(\frac{\ln(K/v) - t\mu}{\sqrt{t\sigma}}\right) \right)$$
BARRIER OPTION CONT.

- **Barrier option** (up-and-in) has time \( s < t \) where, given \( b \), the right to exercise option is conditional on \( P(s) > b \); so

\[
P(s) = ve^X, \quad X \sim \text{Normal}(\mu s, \sigma^2 s),
\]
\[
P(t) = ve^Xe^Y, \quad Y \sim \text{Normal}(\mu(t-s), \sigma^2(t-s)).
\]

Payoff for the option \( R \) is \( R = I(ve^X > b)(ve^{x+y} - K)^+ \), so the expected value for the barrier option is

\[
E[R] = \int_{-\infty}^{\infty} I(ve^x > b)
\]
\[
\int_{-\infty}^{\infty} (ve^{x+y} - K)^+ \frac{e^{-(y-\mu(t-s))^2/2(t-s)\sigma^2}}{\sigma \sqrt{2(t-s)\pi}} dy \frac{e^{-(x-\mu s)^2/2s\sigma^2}}{\sigma \sqrt{2s\pi}} dx.
\]

- Raw simulation: given \( K, b, v, s, t \), each run computes

\( X \sim \text{Normal}(s\mu, s\sigma^2), \)
\( Y \sim \text{Normal}((t-s)\mu, (t-s)\sigma^2), \)

and computes RV

\( R = I(ve^X > b)(ve^{X+Y} - K)^+ \).
• Conditional simulation uses

\[
E[R] = E[R|\text{ } ve^X > b]P\{\text{ } ve^X > b\} + E[R|\text{ } ve^X \leq b]P\{\text{ } ve^X \leq b\}
\]

\[
= E[R|\text{ } ve^X > b]P\{\text{ } ve^X > b\}
\]

\[
= E[R|\text{ } ve^X > b]\left(1 - \Phi\left(\frac{\ln(b/v) - s\mu}{\sigma\sqrt{s}}\right)\right);
\]

in integral form

\[
E[R] = \int_{\ln(b/v)}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2t\sigma^2}}}{\sigma\sqrt{2t\pi}} \int_{-\infty}^{\infty} \left(ve^x + y - K\right)^+ \frac{e^{-\frac{(y-\mu(t-s))^2}{2(t-s)\sigma^2}}}{\sigma\sqrt{2(t-s)\pi}} dy dx
\]

\[
= \int_{\ln(b/v)}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2s\sigma^2}}}{\sigma\sqrt{2s\pi}} \int_{\ln(K_{ve^X})}^{\infty} \left(ve^x + y - K\right) \frac{e^{-\frac{(y-\mu(t-s))^2}{2(t-s)\sigma^2}}}{\sigma\sqrt{2(t-s)\pi}} dy dx.
\]

Each run computes

\[X \sim Normal(s\mu, s\sigma^2), \text{ conditional on } X > \ln(b/v), \text{ and}\]

\[R = \left(1 - \Phi\left(\frac{\ln(b/v) - s\mu}{\sigma\sqrt{s}}\right)\right) \int_{\ln(K_{ve^X})}^{\infty} \left(ve^x + y - K\right) \frac{e^{-\frac{(y-\mu(t-s))^2}{2(t-s)\sigma^2}}}{\sigma\sqrt{2(t-s)\pi}} dy
\]

\[
= \left(1 - \Phi\left(\frac{\ln(b/v) - s\mu}{\sigma\sqrt{s}}\right)\right)C(K, t - s, ve^X).
\]
Conditional generation of $X$? Set

$$U = \int_{\ln(b/v)}^{X} e^{-(x-\mu s)^2/2\sigma^2} dx / \left(1 - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right),$$
or

$$U = \left( \Phi \left( \frac{X-s\mu}{\sigma\sqrt{s}} \right) - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right) / \left(1 - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right);$$

so

$$X = s\mu + \sigma\sqrt{s} \Phi^{-1} \left( \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) + U \left(1 - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right) \right).$$

Then

$$\theta = E[R] \approx \left(1 - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right) \frac{1}{N} \sum_{i=1}^{N} C(K, t-s, ve^{X_i}).$$

Antithetic Variates?

$C$ is increasing function of $X$, so use $X_i$ combined with

$$X_i^* = s\mu + \sigma\sqrt{s} \Phi^{-1} \left( \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) + (1-U) \left(1 - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right) \right).$$

so

$$\theta \approx \frac{P}{N} \sum_{i=1}^{N} \frac{C(K, t-s, ve^{X_i}) + C(K, t-s, ve^{X_i^*})}{2}.$$

with $P = \left(1 - \Phi \left( \frac{\ln(b/v)-s\mu}{\sigma\sqrt{s}} \right) \right)$. 
**ANTITHETIC PERMUTATIONS**

**Antithetic Permutations**: Suppose an estimate is needed for \( \theta = E[f(v_{l_1}, v_{l_2}, \ldots, v_{l_n})] \), for some \( f \), \( l_1, \ldots, l_n \) is random permutation of \( 1, \ldots, n \), and \( v_1 < \ldots < v_n \).

- Example: \( f \) is value of video poker hand.
- If \( V = (v_{l_1}, \ldots, v_{l_n}) \), define the “antithetic permutation” \( V_a = (v_{l_n}, \ldots, v_{l_1}) \)
- Question: is \( \text{Var}((f(V) + f(V_a))/2) < \text{Var}(f(V)) \)?

\[
\text{Var}\left(\frac{f(V) + f(V_a)}{2}\right) = \frac{1}{4}\left(\text{Var}(f(V)) + \text{Var}(f(V_a)) + 2\text{Cov}(f(V), f(V_a))\right),
\]

so \( f(V) \) and \( f(V_a) \) need to be negatively correlated.

- \( f(v) \) is interchange increasing(decreasing) if \( f(v_1) \geq (\leq) f(v_2) \) with \( v_1 = v_2 \) except for interchange of \( v_i, v_j, \) and \( v_i > (\leq) v_j \).
- Theorem: if \( g \) and \( h \) are both interchange increasing or both interchange decreasing then \( \text{Cov}(g(V), h(V_a)) \leq 0 \).
- Video Poker Example: using \( V = \text{randperm}(1, \ldots, 52) \), if card value is \( c_i = \lceil v_i/4 \rceil + 1 \) and suit is \( s_i = \text{mod}(v_i, 4) + 1 \), then the hand value is (weakly) interchange increasing. Hand 2, dealt from bottom of deck is “antithetic hand”.

\[
N = 100000; \quad [XB \ V \ E] = \text{vidpkr}(N); \quad \text{disp}([XB \ V \ E]),
\]

\[
0.8457 \quad 3.0457 \quad 0.011038
\]

\[
[XB \ V \ E] = \text{vidpkra}(N); \quad \text{disp}([XB \ V \ E]),
\]

\[
0.84839 \quad 1.576 \quad 0.011228
\]