**LINE INTEGRALS**

**Line Integral Along a Curve:** given a parameterized smooth curve \( C = \{ \mathbf{r}(t) \mid t \in [a, b] \} \) and \( f(\mathbf{r}) \) defined on \( C \);

- divide \( C \) into \( n \) subarcs with lengths \( \Delta s_1, \Delta s_2, \ldots, \Delta s_n \) and pick one point \( \mathbf{r}(t_i^*) \) from each subarc;
- definition: the **line integral of \( f \) along \( C \)** is
  \[
  \int_C f(\mathbf{r}) ds = \lim_{\max(\Delta s_i) \to 0} \sum_{i=1}^{n} f(\mathbf{r}(t_i^*)) \Delta s_i,
  \]
  a line integral with respect to arc length;
- formula: the line integral of \( f \) along \( C \) is
  \[
  \int_C f(\mathbf{r}) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.
  \]

Note: nonsmooth \( C \) could be split into smooth pieces.

- Application: suppose \( \rho(\mathbf{r}) \) is density for wire, then mass
  \[
  m = \int_C \rho(\mathbf{r}) ds = \int_a^b \rho(\mathbf{r}(t)) |\mathbf{r}'(t)| dt;
  \]
  then center of mass
  \[
  \bar{\mathbf{r}} = \frac{1}{m} \int_a^b \mathbf{r}(t) \rho(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.
  \]
MORE LINE INTEGRALS

Componentwise Line Integrals: given a parameterized smooth curve $C = \{ \mathbf{r}(t) \mid t \in [a, b] \}$ and $f(\mathbf{r})$ for $C$;

- subdivide $C$ using $a = t_0 < t_1 < \cdots < t_n = b$, let $\Delta r_{ij} = r_j(t_i) - r_j(t_{i-1})$, and pick $\mathbf{r}(t_i^*)$’s from subarcs;
- definition: the line integral of $f$ along $C$ with respect to $r_j$ is
  \[
  \int_C f(\mathbf{r})dr_j = \lim_{\max(\Delta r_{ij}) \to 0} \sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\Delta r_{ij};
  \]
- formula: line integral of $f$ along $C$ with respect to $r_j$
  \[
  \int_C f(\mathbf{r})dr_j = \int_a^b f(\mathbf{r}(t))r_j'(t) \, dt;
  \]
- note: if $C$ is (straight)line from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ use
  \[
  \mathbf{r}(t) = \mathbf{r}(a) + t(\mathbf{r}(b) - \mathbf{r}(a)), \quad t \in [0, 1];
  \]
- orientation sometimes matters: if $-C$ denotes curve from $\mathbf{r}(b)$ to $\mathbf{r}(a)$ (backwards along $C$)
  \[
  \int_{-C} f(\mathbf{r})dr_j = \int_a^b f(\mathbf{r}(t))r_j'(t) \, dt
  = - \int_a^b f(\mathbf{r}(t))r_j'(t) \, dt = - \int_C f(\mathbf{r})dr_j;
  \]
  but
  \[
  \int_{-C} f(\mathbf{r})ds = \int_C f(\mathbf{r})ds.
  \]
EVEN MORE LINE INTEGRALS

Vector Field Line Integrals: given
a parameterized smooth curve \( C = \{ r(t) \mid t \in [a, b] \} \)
and a vector field \( \mathbf{F}(r) = < P(r), Q(r), R(r) > \) for \( C \);

- subdivide \( C \) using \( a = t_0 < t_1 < \cdots < t_n = b \);
- if \( \mathbf{F} \) is a force field, then the work done (by \( \mathbf{F} \)) moving a particle from \( r(t_{i-1}) \) to \( r(t_i) \) along \( C \) is
  \[
  W_i \approx \mathbf{F}(r(t_i^*)) \cdot \mathbf{T}(r(t_i^*)) \Delta s_i,
  \]
  where \( \Delta s_i \) is arc length from \( r(t_{i-1}) \) to \( r(t_i) \) along \( C \),
  \( \mathbf{T}(r) \) is unit tangent vector to \( C \) at \( r \), and \( t_i^* \in [t_{i-1}, t_i] \);
- work done moving a particle from \( a \) to \( b \) along \( C \) is
  \[
  W = \lim_{\max(\Delta s_i) \to 0} \sum_{i=1}^{n} W_i = \int_{C} \mathbf{F}(r) \cdot \mathbf{T}(r) \, ds
  \]
  formula: for work done \( a \) to \( b \) along \( C \) is
  \[
  W = \int_{C} \mathbf{F}(r) \cdot \mathbf{T}(r) \, ds = \int_{a}^{b} \mathbf{F}(r(t)) \cdot \frac{r'(t)}{|r'(t)|} |r'(t)| \, dt
  \]
  \[
  = \int_{a}^{b} \mathbf{F}(r(t)) \cdot r'(t) \, dt;
  \]
- definition: the line integral of \( \mathbf{F} \) along \( C \) is
  \[
  \int_{C} \mathbf{F}(r) \cdot dr = \int_{a}^{b} \mathbf{F}(r(t)) \cdot r'(t) \, dt = \int_{C} \mathbf{F} \cdot \mathbf{T} \, ds.
  \]